UNIVERSITY OF CAMBRIDGE

Soft Hair and Subregions in Quantum Gravity

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This thesis is submitted for the degree of Doctor of Philosophy

July 2020

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee. The work presented in Chapters 2 to 5 is adapted from four papers of which I was the sole author [103, 104, 106, 107].

Josh Kirklin

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In quantum gravity and field theory, large gauge transformations lead to novel degrees of freedom living at the boundary. In the presence of a black hole they are called 'soft hair', and it has been suggested that they go some way towards answering the black hole information problem. More generally, they are known as 'edge modes', and play a key role in the structure of the quantum state. Another approach to black hole information involves entanglement, and a key theme of this thesis is to explore aspects of the relationship between edge modes and entanglement.

We first establish a thermodynamical interpretation of soft hair by deriving generalisations of the laws of black hole mechanics. These laws lead to a natural definition of an entropy density on the black hole horizon, and reveal that soft hairs at neighbouring points are in thermal contact with one another.

There are boundary ambiguities in the traditional construction of phase spaces in field theory, and resolving these ambiguities is a step that must be taken before one can fully understand edge modes and soft hair. We provide two possible approaches to such a resolution.

The first approach applies to theories without gravity, and involves a direct evaluation of the Poisson structure from a semiclassical path integral. This is then inverted to give the symplectic structure.

The second approach applies in holographic theories of quantum gravity. We show how one can recover the symplectic structure in a bulk subregion by measuring an object known as 'Uhlmann holonomy' on the boundary, which is a generalisation of Berry phase. The Uhlmann holonomy is actually a direct measure of the entanglement in the quantum state, and so this provides a connection between edge modes and entanglement.

In the final part of the thesis we study Uhlmann phase more generally, showing that it may be computed with a holographic path integral in a *generic* system, so long as the state of the system involves a sufficient degree of entanglement. This suggests that there are deep connections between Uhlmann holonomy, entanglement and holography.

Acknowledgements

I will always be grateful for the advice and support of my supervisor Malcolm Perry. The discussions we have had over the past few years have profoundly shaped my perspective on theoretical physics, and have provided the sparks that led to most of the results in this thesis.

I am also thankful for the numerous helpful conversations I have had with other excellent physicists during my PhD. In particular, thank you to Joan Camps, Maciej Dunajski, Harvey Reall, Gábor Sárosi, David Skinner, Andy Strominger and Aron Wall, and I should also mention William Donnelly, Nick Dorey, Alex Maloney and Nick Manton. Although in some cases we have only interacted once, that interaction has had an important influence on the research in this thesis, and my development as a physicist.

Special thanks are due to my examiners Jonathan Oppenheim and Aron Wall, whose numerous comments and suggestions led to many improvements in the final version of this thesis.

I am deeply appreciative of the hospitality of the physics department at Harvard University, where I spent several months in my second year. I thoroughly enjoyed my time there.

There is no way this thesis could ever have been produced without the companionship of my fellow students in Pavilion B (many of whom are no longer students). So, thank you to Alex Abbott, Muntazir Abidi, Fred Alford, Alec Barns-Graham, Roland Bittleston, Theodor Bjorkmo, Charles Board, Philip Boyle Smith, Phil Clarke, Will Cook, Laurence Cooper, Sam Crew, Toby Crisford, Robin Croft, Joseph Davighi, James Delaney, Amelia Drew, Felicity Eperon, Bogdan Ganchev, Owain Fitz-Gibbon, Tanguy Grall, Sasha Haco, Ward Haddadin, Jonathan Hung, Shayan Iranipour, Chris King, Oliver Leicht, Nakarin Lohitsiri, Maeve Madigan, Hasan Mahmood, Hamed Masaood, João Melo, Nathan Procter, Jonathan Rawlinson, Miren Radia, Roxana Rosca-Mead, Jakub Supeł, Carl Turner, Edward Walton, Alice Waterhouse, Jack Williams, Leong Khim Wong, Antoni Woss and Daniel Zhang. Thanks also to Manda Stagg! I am sorry if I have left anyone out who feels they should be here – there are a lot of you!

Thank you to Mary Dai, Pete Linnegan, Sarah Linnegan, Michael Sharrock and Sam Summers. They have been constant friends during my eight years in Cambridge, and without them I wouldn't have been able to start this PhD, let alone finish it.

Finally, I would like to thank my three wonderful sisters Hannah, Sophie and Charlotte, my four marvellous grandparents, and most of all my awe-inspiringly generous and wise Mah and Pah. A son couldn't ask for better parents. Thank you – I love you.

I will now proceed to entangle the entire area.

Almost Cut My Hair

CROSBY, STILLS, NASH & YOUNG

Contents

Chapter 1

Introduction

When Stephen Hawking discovered the black hole information paradox in 1976, he lit a fuse at the heart of theoretical physics, and almost half a century later we are still sorting through the debris.

Black holes were at that time known to obey four rules, in close analogy with the four laws of a thermodynamical system [20]. The entropy S and temperature T were analogous to

$$S \sim \frac{A}{\lambda}, \qquad T \sim \lambda \frac{\kappa}{8\pi G}, \tag{1.1}$$

where A is the surface area of the event horizon, κ is its surface gravity, and λ is some constant. For the analogy to be exact, the black hole would have to radiate energy like any other hot body, and in the classical theory this cannot be true, since, by definition, no causal curve can be traced from the black hole region to an external observer. However, taking quantum effects into account, Hawking famously showed that black holes do in fact radiate, and moreover that they radiate like a black-body at a temperature $T = \kappa/2\pi$ [76]. This equation implies that $\lambda = 4G$, and so the black hole's entropy should be given by $S = A/4G$.

It didn't take long for Hawking to observe the problem with this picture [77]. One considers a situation in which we have some cloud of matter in a pure quantum state, with many independent quantum numbers. One allows the matter to collapse to a black hole, which at late times one should expect to be approximately stationary. The black hole uniqueness and no-hair theorems [119] tell us that that this black hole is classified by only very few numbers: its mass, angular momentum, and charge. The black hole Hawking-radiates quickly enough that it must eventually lose all energy and evaporate in finite time according to an observer. The spectrum of radiation is a probabilistic one parametrised only by the quantum numbers of the black hole.

It would seem that the degrees of freedom that entered the black hole must be present in the radiation, but the limited parametrisation of the Hawking radiation cannot contain all of these degrees of freedom.

The conclusion one is led to draw is that, after the black hole has evaporated, some degrees of freedom have completely disappeared! This appears to violate the postulate of unitary evolution in quantum mechanics, and this is (one version of) the black hole information paradox. There have been numerous attempts to resolve it – far too many to cover all of them here. Suffice it to say that, although we have gotten closer, a consensus for what exactly is going on has thus far evaded the theoretical physics community. However, one majority view seems to be that unitarity is paramount, and so must be preserved by whatever argument fixes the paradox. This is the view we will take here.

There are then two subtly distinct questions that the paradox raises.

1. Before the black hole has evaporated, where does it store the information about the objects that went into it? In other words, what are the microscopic degrees of freedom that realise the entropy (1.1)?
2. When the black hole is evaporating, how does that information get out?

It may be that the answers to these questions ultimately depend upon whichever fundamental theory of quantum gravity describes reality, and we still don't know what this theory is. However, if one is optimistic, one may hope to be able to answer them using only the semiclassical toolkit that was used to discover the paradox in the first place. This optimism seems most likely to pan out for the first question, since to address it we never need to enter the extreme regime that applies to the black hole near the end of evaporation.

This thesis is motivated in large part by a recent proposal of an answer to the first question, based on the fact that quantum gravity has less gauge symmetry than we used to think. It is possible to construct a field transformation which looks everywhere locally like a gauge transformation, but whose combined global effect is a physical change of state. Such transformations are called 'large gauge transformations' (LGT), and generally come in two varieties. The first variety is topological in nature, and applies when spacetime contains non-contractible loops; we will not be concerned with these kinds of LGTs. Instead, we will focus on the second variety, which applies whenever we are considering a spacetime with a boundary. The boundary can be either asymptotic or finite, and a gauge transformation will be large if the gauge parameter is non-trivial at the boundary (the precise meaning of 'non-trivial' will vary from theory to theory).

The relevance of LGTs is easiest to understand from a Hamiltonian point of view, which makes it easy to see which degrees of freedom are real, and which are not. A very useful formulation of the Hamiltonian treatment of covariant field theories is known as the 'covariant phase space' formalism. We will make frequent use this formalism throughout the thesis, so we will provide a review of it in Section 1.1.

The existence of LGTs has had a profound influence on the study of quantum gravity. For example, arguably the first prototype of AdS/CFT was provided by Brown and Henneaux when they discovered that the algebra of LGTs of 3D gravity with a negative cosmological constant contains a pair of Virasoro subalgebras [39]. More generally, the study of LGTs on the bulk side in holographic theories allows one to understand the symmetries of the boundary theory.

If we want to understand the observable physics of a black hole, the natural part of spacetime to consider is the exterior of the black hole. This has boundaries at the event horizon and at asymptotic infinity, so LGTs will be relevant.

When the black hole uniqueness and no-hair theorems were proved, it was assumed that any two spacetimes related by a gauge transformation were physically equivalent. This was a perfectly natural assumption to make at the time, but in the light of the existence of LGTs it must be revised. Given any black hole spacetime, the action of an LGT will produce a new black hole spacetime, physically inequivalent to the first. Thus, there will be many more degrees of freedom labelling the state of a black hole, beyond just mass, angular momentum and charge. Their existence represents a significant hole in the argument for the information paradox outlined above, and this was the essential observation of [80], which gave them the collective name 'soft hair'. The proposal of that paper was that these degrees of freedom may have sufficient information capacity to account for the black hole entropy. In [71, 72], substantial evidence was provided for this proposal. Those papers showed that the charge algebra generating LGTs in a four-dimensional asymptotically flat black hole spacetime contains a pair of Virasoro subalgebras, and that these have just the right central charges to reproduce the black hole entropy via the Cardy formula [43].

If one considers the more general case of LGTs acting in an aribitrary subregion of spacetime, one finds that there are corresponding degrees of freedom living on the boundary of the subregion. These degrees of freedom are known as 'edge modes', and are the generalisation of soft hair to the subregion.

An alternative proposal for understanding black hole information comes from the realisation that the Bekenstein-Hawking entropy of a black hole [26, 78] can be attributed to entanglement between degrees of freedom on either side of the horizon [33, 137]. This is inspired by a physical picture in which Hawking radiation corresponds to the production of pairs of entangled particles near the horizon. One particle falls in to the black hole, while the other escapes, and over time this results in a large build up of entanglement. This entanglement can be measured by computing a quantity known as the entanglement entropy, and the idea is that the entanglement entropy should be equivalent to the Bekenstein-Hawking entropy. Thus, the black hole information should be understood as being stored in the entanglement.

In holography, this was vastly generalised to the Hubeney-Rangamani-Ryu-Takayanagi (HRT) formula [95, 128, 129], which associates the areas of a large class of bulk surfaces with entanglement entropies of appropriate subsystems. This formula applies even in the absence of a black hole, implying that entanglement plays a key role in the geometry of a generic bulk state.

A theme of this thesis is to explore the connections between the soft hair and entanglement approaches to black hole information. If these approaches are compatible with each other, then the soft hair must somehow describe the configuration of the entanglement of the black hole. More generally, edge modes should reflect entanglement in the structure of the bulk state in quantum gravity, and we would like to understand how exactly this happens.

1.1 Covariant phase space

The Hamiltonian description of any classical physical theory consists of a phase space equipped with a symplectic structure, and a Hamiltonian function. The former is a specification of all the degrees of freedom in the theory, while the latter describes how these degrees of freedom evolve over time. Such a clear split between these two components is very useful when quantising the theory, as one may separately consider the quantum counterparts of each. The phase space is replaced by a Hilbert space, while the Hamiltonian function is replaced with a Hamiltonian operator.

This split between kinematics and dynamics is completely absent from the Lagrangian description of a classical theory. On the other hand, the Lagrangian approach has the advantage that one may often employ it in a manifestly covariant manner, which makes the symmetries of the theory easier to understand. However, the quantisation of a theory starting from its Lagrangian description is not (in general) a very well understood procedure, compared to starting from a Hamiltonian description. One must almost always first carry out some kind of Legendre transformation, in order to convert the Lagrangian description to a Hamiltonian one, and then proceed from there. One may argue that the Lagrangian path integral sidesteps this conversion. But such a path integral is usually only well-defined if we view it as an approximation to a Hamiltonian path integral.[1]

It is still a relatively widely held misconception that in the course of such a conversion one must discard the covariance that makes the Lagrangian approach so attractive. This is most apparent in the canonical approach, the idea there being that one must take a snapshot of the physical system at some fixed time (which breaks covariance in the first instance), and then identify pairs of canonical conjugate variables in that snapshot, making a clear distinction between generalised coordinates and momenta (which breaks covariance in the second instance). One then computes the Hamiltonian in terms of these variables.

There is a different approach one can take. A point in the canonical phase space is the specification of a value for each coordinate and momentum at a fixed time. But the existence of the Hamiltonian implies that if one specifies values for the coordinates and momenta, one obtains a unique solution to the equations of motion (assuming the Cauchy problem is well defined). Similarly, given a solution to the equations of motion, one may deduce the values of the canonical variables at any moment in time. This means that there is a bijection between the canonical phase space and the space of solutions to the equations of motion. One may pullback the canonical symplectic structure to the space of solutions, which then makes the space of solutions a symplectic space isomorphic to the canonical phase space. This construction of the space of solutions and its symplectic structure is independent of the snapshot in time necessary for the canonical construction, and of the splitting between coordinates and momenta. Therefore, it is once more manifestly covariant. For this reason, the space of solutions is commonly known as the covariant phase space.

In the case of field theory, the covariant phase space formalism has its roots in [30, 55, 56, 124], but was solidified in its modern form by [49, 50, 161]. It has since been explored in the work of [17, 21, 40, 75, 92, 96, 111, 117, 155] and many others. The formalism has found many applications, and recently it has been used to investigate symmetries of black hole spacetimes

[1] There are of course exceptions to this statement – sometimes the Lagrangian path integral can be computed exactly without any need for a Hamiltonian description. This is the case for example in some supersymmetric theories. But, in the general case, it is easier to get something resembling a mathematically well-defined path integral with a Hamiltonian approach.

and aspects of the black hole information problem [71, 80, 81]. It is also relevant in holography, where the covariant phase space symplectic structure plays the role of the bulk dual to a natural symplectic structure on the space of boundary sources [28]. Further work in that context [27] has investigated the relation between this symplectic structure and the volume of an extremal bulk slice, in particular revealing a connection to the complexity-volume conjecture [138, 144].

We hope that the rest of this section will serve as a useful introduction to and review of the formalism. For more extensive reviews, see [48, 88].

1.1.1 Phase space, Hamiltonian flow and Poisson brackets

A phase space is a symplectic manifold, i.e. a differential manifold $\mathcal{P}$ equipped with a closed 2-form Ω known as the symplectic form or symplectic structure. Ω must be non-degenerate, which means that if we view Ω as a linear map $\mathrm{T}\mathcal{P} \to \mathrm{T}^*\mathcal{P}$, it must have the property that $\Omega(V) = 0$ if and only if $V = 0$. This implies that the inverse map $-\Pi : \mathrm{T}^*\mathcal{P} \to \mathrm{T}\mathcal{P}$ exists, and we can view this map as an antisymmetric bivector in $\mathrm{T}\mathcal{P} \otimes \mathrm{T}\mathcal{P}$. Π is known as the Poisson structure.

All of the points in phase space correspond to a single classical configuration of a system. A vector in $\mathrm{T}\mathcal{P}$ therefore corresponds to an infinitesimal transformation of the system.

A symplectomorphism (also known as a canonical transformation) is a diffeomorphism of $\mathcal{P}$ that preserves Ω. Consider an infinitesimal diffeomorphism characterised by a vector field ξ. For this to be a symplectomorphism, the Lie derivative of Ω with respect to ξ must vanish, so by Cartan's formula we have

$$0 = \mathcal{L}_\xi \Omega = \iota_\xi \,\mathrm{d}\Omega + \mathrm{d}(\iota_\xi \Omega)\,. \tag{1.2}$$

Here $\mathcal{L}_\xi$ denotes the Lie derivative with repect to ξ, and ι_ξ denotes the interior product (i.e. contraction) with ξ. Since Ω is closed, the first term on the right-hand side is zero. So ξ generates a symplectomorphism if and only if $\iota_\xi \Omega$ is closed. Such vector fields are called symplectic.

If $\iota_\xi \Omega$ is also exact, then the vector field ξ is called Hamiltonian, and the diffeomorphism it generates is called a Hamiltonian symplectomorphism. In this case, we may write $\iota_\xi \Omega = -\,\mathrm{d}f$ for some function on phase space f, which is known as the Hamiltonian which generates the flow along ξ. Conversely, given any function f, we can define ξ_f via $\iota_{\xi_f}\Omega = -\,\mathrm{d}f$, and ξ_f will always exist because of the non-degeneracy of Ω.

We can then define the Poisson bracket of two functions f, g on phase space via

$$\{f, g\} = \Pi(f, g) = \iota_{\xi_f} \iota_{\xi_g} \Omega = -\xi_f(g) = \xi_g(f). \tag{1.3}$$

One may show that

$$\iota_{[\xi_f,\xi_g]}\Omega = \mathrm{d}(\{f,g\}) \,, \tag{1.4}$$

so the symplectomorphism generated by $\{f,g\}$ is $\xi_{\{f,g\}} = -[\xi_f,\xi_g]$.

To finish defining the theory, we pick a special function H on phase space called *the* Hamiltonian. The time evolution of any function f is then determined by Hamilton's equations

$$\dot{f} = \xi_H(f) = \{f,H\}. \tag{1.5}$$

At least locally in phase space, we may write down an action whose variational principle gives this equation of motion. First, we pick a 1-form Θ such that $\Omega = \mathrm{d}\Theta$ (this cannot in general be done globally). Then the Hamiltonian action of a curve C in phase space is defined as

$$S = \int_C (\Theta - H\,\mathrm{d}t), \tag{1.6}$$

where t is a parameter along the curve. Subject to the endpoints of C being fixed, the curves which extremise this action satisfy Hamilton's equations. By this we mean that, if $x(t)$ is the point on C at time t, Hamilton's equations hold for any function $f(x)$ if we evaluate both sides at $x(t)$.

The objective of quantisation is to find a Hilbert space $\mathcal{H}$, and a map from functions f on phase space to operators O_f acting on $\mathcal{H}$, such that the Dirac relation holds

$$O_{\{f,g\}} = \frac{i}{\hbar}[O_f,O_g]. \tag{1.7}$$

(Generally this can only be made to hold at leading order in small $\hbar$.) There are various ways to do this, and we will not comment further on them here.

1.1.2 Pre-phase space and gauge reduction

Consider the case where the form Ω is not non-degenerate. Then we are not dealing with a phase space, but instead a *pre*-phase space, or alternatively a presymplectic manifold, on which Ω is known as the presymplectic form. This turns out to be the natural geometric setting in which to consider a theory with gauge symmetries. To be more precise, if V is a non-zero vector field on $\mathcal{P}$ such that $\iota_V\Omega = 0$, then we cannot hope to find an operator in the quantum theory which corresponds to this transformation. Consequently, we will never be able to measure it, and so we should view V as a gauge transformation.

Suppose V_1, V_2 are two such vector fields, i.e.

$$\iota_{V_1}\Omega = \iota_{V_2}\Omega = 0. \tag{1.8}$$

Then we have

$$\iota_{[V_1,V_2]}\Omega = \mathcal{L}_{V_1}(\underbrace{\iota_{V_2}\Omega}_{=0}) - \iota_{V_2}(\mathcal{L}_{V_1}\Omega) = \iota_{V_1}\mathrm{d}(\underbrace{\iota_{V_1}\Omega}_{=0}) = 0. \tag{1.9}$$

(The first equality is just the product rule.) Thus, the commutator of two gauge transformations is itself a gauge transformation. By Frobenius' theorem, the set of all such vectors V consists of tangent vectors to the leaves of a regular foliation $\mathcal{F}$ of $\mathcal{P}$. Each leaf $\sigma \in \mathcal{F}$ consists of all the possible gauge-equivalent versions of a single physical state. It is clear then that we can think of σ itself as the physical state, and that we should view $\mathcal{F}$ as the true physical phase space. In constructing $\mathcal{F}$, we have 'quotiented by the degenerate directions of Ω'.[2]

Any observable we define must be gauge-invariant. This means that all observable functions f on $\mathcal{P}$ must be constant on the leaves of $\mathcal{F}$. Each such function then unambiguously maps to a single function on $\mathcal{F}$. Note that any function f satisfying $\iota_\xi\Omega = -\mathrm{d}f$ for some ξ is automatically gauge-invariant, since $V(f) = \iota_\xi\iota_V\Omega = 0$.

Because Ω is normal to the leaves of $\mathcal{F}$, there is a unique 2-form $\tilde{\Omega}$ on $\mathcal{F}$ such that Ω is the pullback of $\tilde{\Omega}$ through the natural map $\mathcal{P} \to \mathcal{F}$. The pullback commutes with the exterior derivative, so $\tilde{\Omega}$ is closed, and moreover it is non-degenerate because we have already quotiented by all degenerate directions. Thus $\tilde{\Omega}$ is a symplectic form on $\mathcal{F}$, and so $\mathcal{F}$ is a genuine symplectic manifold.[3]

This route from pre-phase space to phase space is known as gauge reduction, and in principle is a necessary step before quantisation. However, in practice it is often much easier to only deal with the pre-phase space. One just has to make sure that one knows what the gauge transformations are, and that one only considers gauge-invariant observables.

1.1.3 Space of solutions in field theory

Consider a theory of fields ϕ in a spacetime $\mathcal{M}$. Let $\mathcal{C}$ be the space of all possible field configurations; typically this is the space of sections of some bundle over $\mathcal{M}$.

For each possible confuguration in $\mathcal{C}$, one can compute the action

$$S = \int_{\mathcal{M}} L + S_{\partial\mathcal{M}}. \tag{1.10}$$

[2] In the quantum theory, this can be understood by introducing ghost fields, and it is possible to show that this quotienting is in this sense equivalent to the BRST formalism (see for example [89]).

[3] We will assume it is a manifold, but really in the general case it can be a more complicated kind of space, such as an algebraic variety.

Here $L = L[\phi]$ is the Lagrangian density of the theory, i.e. a top form on spacetime which depends locally on the fields ϕ and their derivatives. The contribution $S_{\partial\mathcal{M}}$ is a boundary term, i.e. it depends only on the fields and their derivatives at the boundary.

Consider an infinitesimal change $\phi \to \phi + \delta\phi$ in the field configuration. $\delta\phi$ may be viewed as a vector on configuration space $\mathcal{C}$. Under this change, by using the product rule it is always possible to write the linear order change in the Lagrangian density in the form

$$\delta L = L[\phi + \delta\phi] - L[\phi] = E[\phi] \cdot \delta\phi + \mathrm{d}(\theta[\phi, \delta\phi])\,. \tag{1.11}$$

Here E and θ depend locally on ϕ, and θ also depends linearly and locally on $\delta\phi$. The $\cdot$ denotes a sum over all the different fields and their indices. The action changes by

$$\delta S = \int_{\mathcal{M}} E[\phi] \cdot \delta\phi + \int_{\partial\mathcal{M}} \theta[\phi, \delta\phi] + \delta S_{\partial\mathcal{M}}. \tag{1.12}$$

In order for the variational principle to be well-defined, one picks the boundary term $S_{\partial\mathcal{M}}$ such that the latter two terms above cancel, i.e. $\int_{\partial\mathcal{M}} \theta + \delta S_{\partial\mathcal{M}} = 0$. This often requires the use of supplementary boundary conditions. Then when these boundary conditions are obeyed we have

$$\delta S = \int_{\mathcal{M}} E[\phi] \cdot \delta\phi. \tag{1.13}$$

To extremise the action for any arbitrary $\delta\phi$ we must therefore have $E[\phi] = 0$, and these are the equations of motion.

The space of field configurations which obey the boundary conditions and equations of motion is called the space of solutions, or covariant phase space. Let us label it $\mathcal{P}$. From now on, we will assume that all field configurations and variations are on-shell, meaning $\phi \in \mathcal{P}$ and $\delta\phi \in \mathrm{T}\mathcal{P}$. On-shell, we have

$$\delta L[\phi] = \mathrm{d}(\theta[\phi, \delta\phi])\,. \tag{1.14}$$

1.1.4 The (pre)symplectic form

Let Σ be a Cauchy surface for $\mathcal{M}$, and let

$$\Theta[\phi, \delta\phi] = \int_{\Sigma} \theta[\phi, \delta\phi]. \tag{1.15}$$

Since Θ depends linearly on the vector $\delta\phi$, it is really a 1-form on $\mathcal{P}$.

Let us use the symbol δ to denote the exterior derivative on $\mathcal{P}$, in order to distinguish it from the spacetime exterior derivative d. Note that when δ appears in the vector $\delta\phi$ it is *not* an exterior derivative. In general, δ can also mean the linear order variation in a quantity after the change $\phi \to \phi + \delta\phi$. This notation may be confusing at first, but it is widely used.

Then we define the (pre)symplectic form on $\mathcal{P}$ as the exterior derivative of Θ,

$$\Omega = \delta\Theta. \tag{1.16}$$

This is a presymplectic form if there are gauge symmetries, but if there are no gauge symmetries then it is a symplectic form. In either case, we will for simplicity often just refer to Ω as 'the symplectic form'.

It is useful to write down the components of Ω in terms of two field variations $\delta_1\phi, \delta_2\phi$. Since these are vector fields, we can compute their Lie bracket $\delta_{12}\phi = [\delta_1\phi, \delta_2\phi]$. Then by the definition of the exterior derivative we have

$$\Omega[\phi, \delta_1\phi, \delta_2\phi] = \delta_1\Theta[\phi, \delta_2\phi] - \delta_2\Theta[\phi, \delta_1\phi] - \Theta[\phi, \delta_{12}\phi] = \int_\Sigma \omega[\phi, \delta_1\phi, \delta_2\phi], \tag{1.17}$$

where

$$\omega[\phi, \delta_1\phi, \delta_2\phi] = \delta_1\theta[\phi, \delta_2\phi] - \delta_2\theta[\phi, \delta_1\phi] - \theta[\phi, \delta_{12}\phi]. \tag{1.18}$$

1.1.5 Example: Maxwell field

Let us carry out the construction described above for the simple case of a pure Maxwell gauge field $\mathcal{A}$ with field strength $\mathcal{F} = \mathrm{d}\mathcal{A}$. The Lagrangian density is

$$L = \frac{1}{2}\mathcal{F} \wedge *\mathcal{F}. \tag{1.19}$$

After a field variation $\mathcal{A} \to \mathcal{A} + \delta\mathcal{A}$, the Lagrangian density changes at linear order by

$$\begin{aligned} \delta L &= \frac{1}{2}\mathcal{F} \wedge *\delta\mathcal{F} + \frac{1}{2}\delta\mathcal{F} \wedge *\mathcal{F} && (1.20) \\ &= \mathrm{d}(\delta\mathcal{A}) \wedge *\mathcal{F} && (1.21) \\ &= \mathrm{d}(\delta\mathcal{A} \wedge *\mathcal{F}) + \delta\mathcal{A} \wedge \mathrm{d} * \mathcal{F}\,. && (1.22) \end{aligned}$$

The equations of motion are $\mathrm{d} * \mathcal{F} = 0$, so we can write

$$\theta = \delta\mathcal{A} \wedge *\mathcal{F}. \tag{1.23}$$

Integrating this over a Cauchy surface, and taking a field space exterior derivative, one obtains the symplectic form:

$$\Omega[\mathcal{A}, \delta_1\mathcal{A}, \delta_2\mathcal{A}] = \int_\Sigma \Big(\delta_2\mathcal{A} \wedge *\delta_1\mathcal{F} - \delta_1\mathcal{A} \wedge *\delta_2\mathcal{F}\Big). \tag{1.24}$$

1.1.6 Gauge symmetries

Now that we have a pre-phase space $\mathcal{P}$ with a presymplectic form Ω, we are ready to start discussing gauge symmetries. We will consider local infinitesimal gauge symmetries, i.e. those which depend linearly and locally on some arbitrary spacetime-dependent parameter. This includes for example Maxwell gauge transformations $A \to A + \mathrm{d}\lambda$, in which case the parameter is the function λ, and spacetime diffeomorphisms $\phi \to \phi + \mathcal{L}_\xi \phi$, in which case the parameter is the vector field ξ. Note that not all of these will be 'true' gauge symmetries that leave the physical state unchanged – some will correspond to large gauge transformations. The pair $\mathcal{P}, \Omega$ allows us to determine which are large and which are not.

Let us consider a theory with diffeomorphism gauge symmetry. We will consider some other types of gauge transformations, for which similar methods apply, later in the thesis. Suppose we transform the fields according to $\delta_\xi \phi = \mathcal{L}_\xi \phi$. In a diffeomorphism-invariant theory, the Lagrangian density must be covariantly constructed from the fields, which implies that under this change we will have $\delta_\xi L = \mathcal{L}_\xi L$. Note that this is far from a general property of any theory. For example, the presence of non-trivial background fields would invalidate this equation.

Since L is a top form, we have $\mathcal{L}_\xi L = \mathrm{d}(\iota_\xi L)$. Substituting this into (1.14), we obtain

$$\mathrm{d}(\iota_\xi L) = \mathcal{L}_\xi L = \mathrm{d}(\theta[\phi, \mathcal{L}_\xi \phi])\,. \tag{1.25}$$

This implies that

$$j_\xi[\phi] = \theta[\phi, \mathcal{L}_\xi \phi] - \iota_\xi L[\phi] \tag{1.26}$$

is closed as a spacetime form.

Actually, a result known as the algebraic Poincaré lemma, and proven for example in [154], implies something stronger. If a closed form depends locally and linearly on a spacetime-dependent parameter ξ, then the algebraic Poincaré lemma implies it is exact. In this case, j_ξ has the right dependence on ξ, so it is exact. This means we can write

$$j_\xi[\phi] = \mathrm{d}(Q_\xi[\phi]) \tag{1.27}$$

for some Q_ξ.

Now consider the field variation $[\delta, \delta_\xi]\phi$ obtained by taking the commutator of an arbitrary field variation $\delta\phi$ with the diffeomorphism $\delta_\xi \phi = \mathcal{L}_\xi \phi$. We have

$$[\delta, \delta_\xi]\phi = \delta(\mathcal{L}_\xi \phi) - \mathcal{L}_\xi(\delta\phi) = \mathcal{L}_{\delta\xi}\phi, \tag{1.28}$$

where the last line follows from the linearity of the Lie derivative. Thus, this variation is actually just a diffeomorphism along $\delta\xi$, so we can write

$$\theta[\phi, [\delta, \delta_\xi]\phi] = j_{\delta\xi}[\phi] + \iota_{\delta\xi} L[\phi]. \tag{1.29}$$

In general ξ can depend on the fields ϕ, so $\delta\xi \neq 0$.

Diffeomorphism-invariance means $\theta[\phi, \delta\phi]$ must be covariantly constructed from the fields and their variations, so we must have

$$\delta_\xi(\theta) = \mathcal{L}_\xi \theta = \mathrm{d}(\iota_\xi \theta) + \iota_\xi \underbrace{\mathrm{d}\theta}_{=\delta L}. \tag{1.30}$$

Combining (1.26), (1.29) and (1.30), one finds

$$\omega[\phi, \delta_\xi\phi, \delta\phi] = \delta_\xi(\theta[\phi, \delta\phi]) - \delta(\theta[\phi, \delta_\xi\phi]) + \theta[\phi, [\delta, \delta_\xi]\phi] \tag{1.31}$$

$$= \mathrm{d}(\iota_\xi\theta) + \iota_\xi \delta L - \delta(j_\xi + \iota_\xi L) + j_{\delta\xi} + \iota_{\delta\xi} L. \tag{1.32}$$

By linearity, $\delta(\iota_\xi L) = \iota_\xi \delta L + \iota_{\delta\xi} L$. So, substituting in (1.27), we find that (1.32) may be written as an exact form

$$\omega[\phi, \delta_\xi\phi, \delta\phi] = \mathrm{d}(-\delta(j_\xi[\phi]) + j_{\delta\xi}[\phi] + \iota_\xi\theta[\phi, \delta\phi]). \tag{1.33}$$

Using (1.27), we see that contracting the vector $\delta_\xi\phi$ into the symplectic structure $\Omega = \int_\Sigma \omega$ gives a boundary integral over $\partial\Sigma$,

$$\Omega[\phi, \delta_\xi\phi, \delta\phi] = -\int_{\partial\Sigma} \Big(\delta(Q_\xi[\phi]) - Q_{\delta\xi}[\phi] - \iota_\xi\theta[\phi, \delta\phi]\Big). \tag{1.34}$$

Recall that $\delta_\xi\phi \neq 0$ is a true gauge symmetry if and only if $\Omega[\phi, \delta_\xi\phi, \delta\phi] = 0$. For this to be satisfied certain components of ξ will need to vanish at $\partial\Sigma$. Otherwise, we have $\Omega[\phi, \delta_\xi, \delta\phi] \neq 0$, and $\delta_\xi\phi$ is instead a large gauge transformation. In the case of diffeomorphisms, LGTs are sometimes called large diffeomorphisms. Similar results apply for other types of gauge transformations.

For certain large diffeomorphsims ξ, the transformation $\delta_\xi\phi$ will be Hamiltonian, i.e. it will satisfy $\Omega[\phi, \delta_\xi\phi, \delta\phi] = -\delta H_\xi$ for some function H_ξ on $\mathcal{P}$ (this will not be true in general). In this case we call the LGT 'integrable'. It is important to determine which LGTs are integrable and which are not. This is because integrable LGTs actually preserve the symplectic structure, and so are symmetries of the classical theory, and will be promoted to symmetries of the quantum theory generated by operators $\hat{H}_\xi$ (unless there are anomalies). Non-integrable LGTs are not symmetries of the theory, but are still worth studying in some cases.

For a class of examples of integrable large diffeomorphisms, one can suppose ξ is tangent to $\partial\Sigma$, and $\delta\xi = 0$. Then both $Q_{\delta\xi}$ and $\iota_\xi\theta$ (when pulled back to $\partial\Sigma$) will vanish. This means only the δQ_ξ piece will remain, and so we can write

$$\Omega[\phi, \delta_\xi\phi, \delta\phi] = -\delta(H_\xi[\phi]) \quad \text{where} \quad H_\xi[\phi] = \int_{\partial\Sigma} Q_\xi[\phi]. \tag{1.35}$$

So this large diffeomorphism is integrable and generated by H_ξ.

We should note that sometimes the boundary conditions that make up the definition of the theory will break local gauge symmetry at $\partial\Sigma$. This means that an arbitrary gauge transformation might not obey the boundary conditions, and so would not actually be a vector that stays within the space of solutions. We should disregard such gauge transformations.

We can summarise some of the above by noting the following heirarchy:

$$\begin{aligned}&\{\text{integrable large gauge transformations}\} \subset \{\text{large gauge transformations}\} \\ &\quad \subset \{\text{boundary condition preserving gauge transformations}\} \subset \{\text{gauge transformations}\}.\end{aligned}$$

1.1.7 Ambiguities

There are a number of ambiguities of varying severity in the formalism we have just described. Let us address these now.

- Recall the action of the theory

  $$S = \int_{\mathcal{M}} L + S_{\partial\mathcal{M}}. \tag{1.36}$$

  If we carry out a redefinition

  $$L \to L + \mathrm{d}X\,, \qquad S_{\partial\mathcal{M}} \to S_{\partial\mathcal{M}} - \int_{\partial\mathcal{M}} X, \tag{1.37}$$

  where $X = X[\phi]$ is some form that depends locally on the fields, then the action does not change, and so neither do the equations of motion. However, under this redefinition we have

  $$\mathrm{d}\theta = \delta L \to \delta L + \mathrm{d}\delta X\,, \tag{1.38}$$

  and so

  $$\Theta = \int_\Sigma \theta \to \int_\Sigma \theta + \delta X = \Theta + \delta\Big(\int_\Sigma X\Big). \tag{1.39}$$

  Fortunately, the symplectic form itself

  $$\Omega = \delta\Theta = \delta\Theta + \underbrace{\delta^2}_{=0}\Big(\int_\Sigma X\Big) \tag{1.40}$$

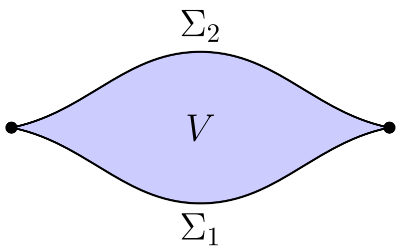


Figure 1.1: The symplectic structure Ω is the same for two Cauchy surfaces Σ_1, Σ_2 if they bound a region V. Note that for this to hold the two surfaces must share a boundary $\partial\Sigma_1 = \partial\Sigma_2$.

does not change. Since the symplectic form is what controls the physics, this does not actually amount to a physical ambiguity.

- The form Q_ξ was given by (1.27). However, that equation only defines Q_ξ up to the addition of an exact form, i.e.

$$Q_\xi \to Q_\xi + \mathrm{d}F_\xi \, . \tag{1.41}$$

Fortunately, whenever we actually use Q_ξ to compute physical quantities, for example in (1.34), we integrate it over $\partial\Sigma$. The integral of an exact form over a boundary always evaluates to zero, which is a simple consequence of Stokes' theorem and the fact that $\partial^2\Sigma = \emptyset$:

$$\int_{\partial\Sigma} \mathrm{d}F_\xi = \int_{\partial(\partial\Sigma)} F_\xi = 0. \tag{1.42}$$

Therefore, again this is not a physical ambiguity.

- There are different choices one could make for the Cauchy surface Σ, and this can lead to different definitions of the symplectic structure $\Omega = \int_\Sigma \omega$. However, we should note that

$$\mathrm{d}\omega = \delta(\mathrm{d}\theta) = \delta(\delta L) = 0, \tag{1.43}$$

since the exterior derivative obeys $\delta^2 = 0$. Thus, ω is a closed form on spacetime, so if Σ_1, Σ_2 bound a region V as shown in Figure 1.1, then we have (picking appropriate orientations for $\Sigma_{1,2}$)

$$\int_{\Sigma_2} \omega - \int_{\Sigma_1} \omega = \int_V \mathrm{d}\omega = 0, \tag{1.44}$$

so Ω will be the same for these two choices of Cauchy surface.

So this amibiguity can only apply if we pick two Cauchy surfaces which do not bound a region in this way. Consider the infinitesimal case in which Σ_2 is related to Σ_1 by a deformation according to a vector field χ. Then we have

$$\int_{\Sigma_2} \omega - \int_{\Sigma_1} \omega = \int_{\Sigma_1} \mathcal{L}_\chi \omega = \int_{\partial\Sigma_1} \iota_\chi \omega. \tag{1.45}$$

So the two symplectic forms will be equivalent if and only if $\iota_\chi \omega$ gives zero when integrated over $\partial\Sigma_1$. Clearly this will be true when χ is tangent to $\partial\Sigma_1$ (in this case $\partial\Sigma_1 = \partial\Sigma_2$ and Σ_1, Σ_2 bound a volume V, so we already knew this). But it can also be true for more general χ, depending on the boundary conditions at $\partial\Sigma$.

Despite these special cases, there can still be significant differences in Ω for different choices of Σ. However, at this point we should not view this as an ambiguity. Rather, the choice of Σ is an input in the definition of the theory. Usually we are well-informed enough about the nature of the theory that we can pick Σ in an appropriate way. For example, we might be doing QFT on a curved Schwarzschild background, and want to understand the physics accessible to an observer on one side of the black hole. Then we would pick Σ to be a partial Cauchy surface for the external region.

- The most crucial ambiguity is the following. The formula $\delta L = \mathrm{d}\theta$ only defines θ up to the addition of a closed form k which is linearly locally dependent on $\delta\phi$. By the results of [154], k is exact, so let $k = \mathrm{d}\alpha$. The corresponding change in ω is given by

$$\omega \to \omega - \mathrm{d}(\delta_1(\alpha[\phi, \delta_2\phi]) - \delta_2(\alpha[\phi, \delta_1\phi]) - \alpha[\phi, \delta_{12}\phi]) \,, \tag{1.46}$$

and the symplectic structure changes by

$$\Omega \to \Omega - \int_{\partial\Sigma} \delta_1(\alpha[\phi, \delta_2\phi]) - \delta_2(\alpha[\phi, \delta_1\phi]) - \alpha[\phi, \delta_{12}\phi]. \tag{1.47}$$

This amounts to a genuine physical change in the theory. The change in (1.45) is actually a special case of this one, with $\alpha = \iota_\chi\theta$. However, it is less clear how one should deal with this ambiguity.

This boundary ambiguity in the symplectic structure will clearly have a large impact on LGTs. For example, different choices of boundary terms will lead to different LGTs being integrable.

In fact, this problem was more or less solved by the authors of [75], who showed that by carefully considering the boundary conditions and boundary term in the action one can fix a unique symplectic form. Moreover, this symplectic form is the 'right' symplectic form, because one can use it to write the action S in Hamiltonian form (1.6).

1.1.8 Phase space of a subregion

Up to now, we have been considering the phase space of the entire field theory. However, it is very interesting to see what happens when one restricts to the degrees of freedom in a proper subregion of spacetime. This is the natural setting for example when one wishes to understand the physics that is accessible to an observer in the exterior of a black hole. More generally, exploration in this direction enables one to understand the way in which the local degrees of freedom are structured in the theory.

A natural and often used definition for the presympletic form of a subregion is

$$\Omega = \int_{\Sigma} \omega, \tag{1.48}$$

where instead of Σ being a Cauchy surface, it is now a *partial* Cauchy surface. This is supposed to be the presymplectic form governing the physics in $D(\Sigma)$, the domain of dependence of Σ. Note that now Ω is definitely a presymplectic form, regardless of whether the theory is a gauge theory or not, because it excludes those degrees of freedom outside of $D(\Sigma)$. In particular, if $\delta\phi$ is a variation of the fields which changes only degrees of freedom outside of $D(\Sigma)$, then it will be a degenerate direction of Ω. (We will still continue to refer to Ω as a 'symplectic form'.)

However, with this definition of the subregion symplectic form, the final ambiguity (1.47) noted in the previous subsection is now much less easy to resolve. Indeed, one no longer has access to a predetermined set of boundary conditions or boundary terms at $\partial\Sigma$, so the method of [75] no longer applies.

This ambiguity is extremely important when one wants to understand LGTs in a subregion. This was the setup for example in [71, 72], where the subregion in question was the exterior of a black hole, and the LGTs had non-trivial action at the event horizon. A particular choice of symplectic form was made in those papers, so that a certain set of LGTs became integrable. This was a necessary step to show that black hole soft hair could account for the entropy. But no a priori motivation was given for the choice.

There are three possible approaches to dealing with the ambiguity. First, one could give up, and say that there is no unambiguous way to define the phase space of a subregion. But we find this unsatisfactory, because there are many physically well-motivated questions one can ask which require such a phase space to exist.[4] Next, one could claim that the subregion phase space does exist, but that it is not enough to just pick Σ, and that one needs to supply some

[4] An example of such a question is: 'How much energy is there in a subregion?'. This question is well-defined in

additional information which resolves the ambiguity. We also find this unsatisfactory, because such information represents a large amount of arbitrary extra input to the theory.[5] Finally, one may guess that there is already a resolution of the ambiguity implicit in the theory, but that the recipe for the symplectic form Ω described above is incomplete. This is the view that we will take in the thesis.

1.2 Overview of thesis

The rest of the thesis explores some problems related to these topics described above. Chapters 2, 3, 4 and 5 are based on material that originally appeared in the papers [104], [107], [106] and [103] respectively. Let us now give a brief overview of these chapters.

In Chapter 2, we discuss LGTs in asymptotically flat space, which are generated by charges defined at asymptotic infinity. No method for unambiguously localising these charges into the interior of spacetime has previously been established. We determine what this method must be, and use it to find localised expressions for the LGT charges. By applying the same principle to the case of a charged black hole spacetime, we find angle-dependent generalisations of the Smarr formula and the first law of black hole mechanics, both of which have important thermodynamical implications. In particular, the presence of a heat current intrinsic to the event horizon is observed.

In the next two chapters, we address the ambiguity in the definition of the subregion phase space noted in Section 1.1.8.

In Chapter 3, we consider the case of a QFT without diffeomorphism symmetry, i.e. a theory without gravity. We provide a resolution of the ambiguity by directly computing the Poisson structure from the path integral, which we invert to find the symplectic structure, showing that it may be written as a contour integral around a partial Cauchy surface. We comment on some implications for gauge symmetry and entanglement.

Then, in Chapter 4, we consider the case of a theory with gravity. In particular, we study a

theories without gravity. In gravitational theories, subregion duality (which will be discussed more in Chapter 4) provides a wealth of questions in the holographic case.

[5] It is possible that there are many different ways to fix the ambiguity that somehow actually end up giving physically equivalent theories. For example, the quantum theories could be unitarily equivalent. However, we are unaware of evidence that this will happen in the general case.

holographic theory of quantum gravity, and find the boundary dual of the symplectic form for the bulk fields in any entanglement wedge. The key ingredient is Uhlmann holonomy, which is a notion of parallel transport of purifications of density matrices based on a maximisation of transition probabilities. Using a replica trick, we compute this holonomy for curves of reduced states in boundary subregions of holographic QFTs at large N, subject to changes of operator insertions on the boundary. We show that the Berry phase along Uhlmann parallel paths may be written as the integral of an Abelian connection whose curvature is the symplectic form of the entanglement wedge. This generalises previous work on holographic Berry curvature [28].

In Chapter 5, we study further the relationship between Uhlmann holonomy and holography. In particular, we obtain a path integral formula for the Uhlmann phase of a generic system. We show that, in a classical limit in which the state of the system is highly entangled, the action for the path integral contains an emergent extra dimension. Thus, despite not making any assumptions about the system in question having any holographic features, we find that the Uhlmann phase necessitates the introduction of an emergent holographic bulk.

Chapter 2

Localisation of Soft Charges, and Thermodynamics of Soft Hair

2.1 Introduction

Consider a theory of fields in an asymptotically flat spacetime $\mathcal{M}$. In the covariant Hamiltonian approach to the analysis of such a theory described in Section 1.1, one must choose a $\partial\Sigma$ at which all Cauchy surfaces Σ must have their boundary, and a set of boundary conditions at $\partial\Sigma$. If $\overline{\mathcal{M}}$ is the conformal compactification[1] of $\mathcal{M}$ (and restricting to the case where $\partial\Sigma$ is connected), then the component of $\partial\Sigma$ at infinity can take one of three possible values:

- Either $\partial\Sigma = i^0$, spacelike infinity, which is the singular point at infinite spacelike distance from all points in $\mathcal{M}$, or
- $\partial\Sigma = \mathscr{I}^+_-$ or $\partial\Sigma = \mathscr{I}^-_+$, the past/future endpoints of future/past null infinity respectively. Future/past null infinity are the unions of all points in $\partial\overline{\mathcal{M}}$ which are the future/past endpoints respectively of null curves originating in $\mathcal{M}$.

Despite their proximity on a Penrose diagram, shown in Figure 2.1, these three choices are not the same.

Historically, the most common decision has been $\partial\Sigma = i^0$ (the most well-known example of this being [11]). In some sense this is not particularly surprising; it is the most obvious immediate choice, especially in the absence of an understanding of the conformal structure of

[1] In some cases a conformal compacitification does not exist, e.g. in odd dimensions greater than 4, in the presence of radiation [93]. In this chapter we focus on the four-dimensional case, for which $\overline{\mathcal{M}}$ does exist.

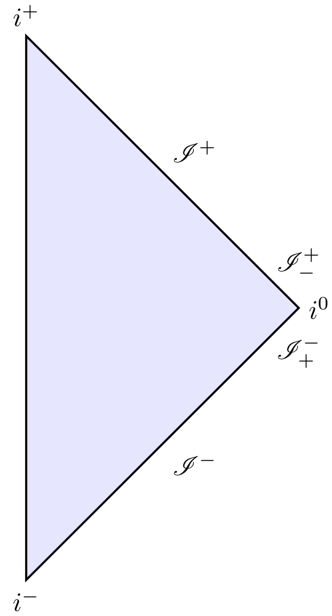


Figure 2.1: The Penrose diagram of asymptotically flat space

asymptotically flat space. However, from the point of view of the scattering problem, it is not the most helpful one. Spatial infinity is completely causally disconnected from the physical spacetime $\mathcal{M}$. In other words, an observer cannot exist at i^0. The only places early-time observations and late-time observations can be made are near $\mathscr{I}^- \cup i^-$ and $\mathscr{I}^+ \cup i^+$ respectively ($i^\pm$ are future/past timelike infinity, defined as the unions of all future/past endpoints of timelike curves from $\mathcal{M}$). Hence it makes the most sense in this context to pick $\partial\Sigma = \mathscr{I}^+_-$ or $\partial\Sigma = \mathscr{I}^-_+$. The classical scattering map can then be concretely realised as a bijection $S : Z^- \to Z^+$, where $Z^\pm$ are the phase spaces obtained by considering $\partial\Sigma = \mathscr{I}^\pm_\mp$ respectively.

It is not immediately clear how the system obtained by choosing $\partial\Sigma = \mathscr{I}^\pm_\mp$ instead of $\partial\Sigma = i^0$ will differ. Certainly they will share most of their features. A fruitful line of research [84–86, 101, 102, 141, 142] has revealed that in gauge and gravity theories, there is at least one quality that the former have which is not shared by the latter.[2] This is the existence of infinitely many more independent and physically significant degrees of freedom associated to large gauge transformations. The charges generating these gauge transformations are known as soft charges, and have deep connections to classical memory effects and quantum mechanical soft theorems. Additionally, one particular scenario in which the quantum scattering problem has been difficult to tackle has been when $\mathcal{M}$ contains a black hole, and the newly observed existence of the soft charges has shed some light on this issue [80, 81]. For reviews on these topics, see [140] or [48].

[2] Although, see [90, 91] for recent evidence otherwise.

A key issue that has not yet been fully resolved is that of the localisation of these charges. One naturally initially derives expressions for the charges in terms of fields at infinity. One also interprets the charges as generating transformations of the fields at infinity. However, it is desirable to find expressions for the charges in terms of fields in the interior. For example if one is interested in the soft charges of a black hole, one would like to express these quantities in terms of fields on its event horizon. In [81], a specific gauge choice was made to extend the gauge transformation under consideration from infinity to the black hole horizon. Another approach has been to consider certain symmetries of the black hole horizon which are analogous to those at infinity, and to derive charges which generate these [60, 61].

The reason we want to be able to do this is as follows. We would like to be able to think of black holes, and other extended objects in spacetime, as having self-contained properties. These properties should depend in an explicit manner on the values of fields local to that object. On the other hand, up to now the soft hair and soft charges of a black hole have only been understood in terms of the fields at infinity, which has obscured the extent to which these are properties of the black hole itself, rather than properties of the rest of the spacetime. Additionally, in a situation in which we might want to consider a spacetime containing more than one black hole, it would be useful to understand the soft hair and soft charges of each black hole individually. Without a localisation procedure, this is impossible.

The first aim of this chapter is to present a simple principle for the localisation of soft charges. Unlike the first approach above, the method is gauge invariant. It serves to provide a relation between the charges at infinity and those at the horizon obtained by the second approach above. It has a simple geometric interpretation.

The second aim of this chapter is with regard to the thermodynamics of black holes [20, 44, 122]. Consider a spacetime containing a black hole. The soft charges of this spacetime provide a notion of the energy, angular momentum, electric charge, etc. of the black hole at each angle on the celestial sphere, and it is natural to try to extend this so that one has a complete thermodynamical system at each angle. Using the localisation technique developed previously, we will generalise the laws of black hole mechanics so that they involve the soft charges. These generalised laws will consequently describe the thermodynamics at each angle, and lead to a natural definition of an angular entropy density. They will also reveal that the system at each angle is not closed, but is in thermal contact with the systems at neighbouring angles. We will find an expression for the resulting angular heat flux. This heat flux can also be viewed as existing on the horizon of the black hole.

It should be pointed out that similar generalised laws of thermodynamics exist in the context of isolated horizons [12–16]. However in those generalisations, the physical meanings of the chosen varied quantities is not always clear. In our case, the varied quantities are explicitly characterised in terms of soft charges.

The chapter is laid out as follows. First, in Section 2.2, we review the Einstein-Maxwell description of isolated electromagnetic gravitational systems, and derive expressions for the soft charges of such systems. Section 2.3 then provides an explanation of our method for localising these expressions to the interior of the system. Next, in Section 2.4, we apply this technique to a stationary black hole. This allows us to obtain generalisations of Smarr's formula and the first law of black hole mechanics. Finally, we will close with some discussion on the results we have obtained, before suggesting future directions.

2.2 Isolated electromagnetic gravitational systems

An electromagnetic gravitational system with metric g_{ab} and electromagnetic potential $\mathcal{A}$ on a 4-dimensional manifold $\mathcal{M}$ is described by the Einstein-Maxwell action $S = S_{\partial\mathcal{M}} + \int_{\mathcal{M}} L$. $S_{\partial\mathcal{M}}$ is a boundary term necessary to make the variational principle well defined, and we will not discuss it in detail here. L is the Lagrangian density 4-form, and is given by[3]

$$L = \frac{1}{16\pi G}\epsilon R + \frac{1}{2e^2}\mathcal{F}\wedge *\mathcal{F}, \tag{2.1}$$

where R is the scalar curvature of g_{ab}, $\epsilon = \sqrt{-g}\,\mathrm{d}^4x$ is the volume form (with $g = \det g_{ab}$), $*$ is its associated Hodge star, and $\mathcal{F} = \mathrm{d}\mathcal{A}$ is the electromagnetic field strength. G and e are coupling constants.

In this section we will analyse this system using the formalism described in Section 1.1. We will disregard the boundary ambiguities, which may be brought under control using [75], and which do not affect the localisation and thermodynamics results of this chapter anyway.

Consider an infinitesimal field variation $g_{ab} \to g_{ab} + \delta g_{ab}$, $\mathcal{A} \to \mathcal{A} + \delta\mathcal{A}$.[4] We will use δ as a shorthand for this variation. To linear order in $\delta g_{ab}, \delta\mathcal{A}$, the corresponding variation of the

[3] We ignore the possibility of any additional matter contributions in this chapter.

[4] We should clarify that, in our notation, δg_{ab} is a tensor whose indices should be raised and lowered with the metric in the standard way. This means that, for example, $\delta g^{ab} = g^{ac}g^{bd}\delta g_{cd}$, which is *not* the variation of the inverse metric. We will instead indicate the variation of the inverse metric by explicitly including parentheses for the variation operator: $\delta(g^{ab})$. We then have $\delta(g^{ab}) = -\delta g^{ab}$.

Lagrangian density is given by $\delta L = \epsilon E^{ab}_{\text{Einstein}} \delta g_{ab} + E_{\text{Maxwell}} \wedge \delta \mathcal{A} + \mathrm{d}\theta$, where

$$E^{ab}_{\text{Einstein}} = -\frac{1}{16\pi G}\left(R^{ab} - \frac{1}{2} g^{ab} R\right) - \frac{1}{2e^2}\left(\mathcal{F}^{ca}\mathcal{F}_c{}^b - \frac{1}{4} g^{ab}\mathcal{F}_{cd}\mathcal{F}^{cd}\right), \tag{2.2}$$

$$E_{\text{Maxwell}} = -\frac{1}{e^2}\,\mathrm{d} * \mathcal{F} \tag{2.3}$$

are the left hand sides of the Einstein and Maxwell field equations $E^{ab}_{\text{Einstein}} = 0, E_{\text{Maxwell}} = 0$ respectively, and

$$\theta[\delta] = \frac{1}{16\pi G}\epsilon_a(\nabla_b \delta g^{ab} - \nabla^a \delta g) + \frac{1}{e^2}\delta\mathcal{A} \wedge *\mathcal{F} \tag{2.4}$$

is the Einstein-Maxwell 'presymplectic potential density'. Here $\delta g = g^{ab}\delta g_{ab} = \delta(\ln g)$, and ϵ_a indicates the volume form with its first index exposed (in this notation, $V^a \epsilon_a = \iota_V \epsilon$ for all vectors V). θ is a 3-form in spacetime, and at the same time a 1-form in field space, since it is a linear functional of the field variation δ.

We see that for any on-shell configuration, the variation of the Lagrangian density is an exact spacetime form $\delta L = \mathrm{d}\theta$. Let δ_1, δ_2 be two on-shell field variations. From the results of Section 1.1, one finds that $\mathrm{d}\omega = 0$, where

$$\omega[\delta_1, \delta_2] = \delta_1(\theta[\delta_2]) - \delta_2(\theta[\delta_1]) - \theta[\delta_{12}] \tag{2.5}$$

$$= \frac{1}{8\pi G}\left[\delta_1(\epsilon_a g^{b[c})\delta_2 \Gamma^{a]}_{bc} - \delta_2(\epsilon_a g^{b[c})\delta_1 \Gamma^{a]}_{bc}\right] - \frac{1}{e^2}\left[\delta_1\mathcal{A} \wedge \delta_2(*\mathcal{F}) - \delta_2\mathcal{A} \wedge \delta_1(*\mathcal{F})\right]. \tag{2.6}$$

We have used the identity $\nabla_b \delta g^{ab} - \nabla^a \delta g = 2g^{b[c}\delta\Gamma^{a]}_{bc}$ in order to write down the compact expression above. ω is the Einstein-Maxwell 'presymplectic structure density'. To obtain the Einstein-Maxwell presymplectic structure Ω, one integrates ω over any Cauchy surface Σ.

2.2.1 Large gauge transformations and soft charges

A general gauge transformation in Einstein-Maxwell theory is the combination of a diffeomorphism, parametrised by a vector field χ, and a Maxwell gauge transformation, parametrised by a function λ. Under this transformation, the metric and gauge potential infinitesimally transform as

$$\begin{aligned} g_{ab} &\to g_{ab} + \mathcal{L}_\chi g_{ab} \\ &= g_{ab} + \nabla_a \chi_b + \nabla_b \chi_a, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathcal{A} &\to \mathcal{A} + \mathcal{L}_\chi \mathcal{A} + \mathrm{d}\lambda \\ &= \mathcal{A} + \iota_\chi \mathcal{F} + \mathrm{d}(\iota_\chi \mathcal{A} + \lambda)\,. \end{aligned} \tag{2.8}$$

Let us use $\delta_{\chi,\lambda}$ to denote this variation. The objective of this section is to find which χ, λ give rise to physical $\delta_{\chi,\lambda}$. By 'physical' we mean that the gauge transformation changes the physical state of the system. These are the large gauge transformations. Non-physical gauge transformations are referred to as small. We can determine which gauge transformations are physical by using the method of Section 1.1.6.

To proceed, we will contract $\delta_{\chi,\lambda}$ into Ω. We could use directly the expression for Ω given in (2.6), but it turns out to be more convenient to return to the form of ω given by (2.5). We shall set $\delta_1 = \delta_{\chi,\lambda}, \delta_2 = \delta$, and evaluate the result term by term.

The first term is $\delta_{\chi,\lambda}(\theta[\delta])$. The only part of $\theta[\delta]$ that transforms non-trivially under a Maxwell gauge transformation is $\delta\mathcal{A} \to \delta\mathcal{A} + \mathrm{d}\delta\lambda$. Hence we have

$$\delta_{\chi,\lambda}(\theta[\delta]) = \mathcal{L}_\chi\theta[\delta] + \frac{1}{e^2}\,\mathrm{d}\delta\lambda \wedge *\mathcal{F} = \mathrm{d}\left(\iota_\chi\theta[\delta] + \frac{1}{e^2}\delta\lambda * \mathcal{F}\right) + \iota_\chi\,\mathrm{d}(\theta[\delta]) \tag{2.9}$$

The second term is $\delta(\theta[\delta_{\chi,\lambda}])$. First note that if we contract $\delta_{\chi,\lambda}$ into $\delta L = \mathrm{d}(\theta[\delta])$, we obtain[5]

$$\mathrm{d}(\iota_\chi L) = \mathrm{d}(\theta[\delta_{\chi,\lambda}])\,, \tag{2.10}$$

implying that $j_{\chi,\lambda} = \theta[\delta_{\chi,\lambda}] - \iota_\chi L$ is a closed 3-form. $j_{\chi,\lambda}$ is the (Hodge dual of) the Noether current associated to this gauge transformation. By the algebraic Poincaré Lemma, we can write $j_{\chi,\lambda} = \mathrm{d}Q_{\chi,\lambda}$ for some 2-form $Q_{\chi,\lambda}$ – the (Hodge dual of) the Noether charge density for this gauge transformation. Indeed, we can take

$$Q_{\chi,\lambda} = \frac{1}{16\pi G} * \mathrm{d}\chi + \frac{1}{e^2}(\iota_\chi\mathcal{A} + \lambda) * \mathcal{F}, \tag{2.11}$$

where for notational simplicity we are using χ to mean both the 1-form $\chi^a g_{ab}\,\mathrm{d}x^b$, and the vector $\chi^a\frac{\partial}{\partial x^a}$. Hence we can write

$$\delta(\theta[\delta_{\chi,\lambda}]) = \mathrm{d}\delta(Q_{\chi,\lambda}) + \delta(\iota_\chi L). \tag{2.12}$$

For the third term $\theta[\delta_{12}]$, it helps to explicitly note what the action of $\delta_{12} = [\delta_{\chi,\lambda}, \delta]$ is. We have

$$\begin{aligned}\delta_{\chi,\lambda}(\delta g_{ab}) - \delta(\delta_{\chi,\lambda}g_{ab}) &= \chi^c\partial_c\delta g_{ab} + \delta g_{ac}\partial_b\chi^c + \delta g_{bc}\partial_a\chi^c - \delta(\chi^c\partial_c g_{ab} + g_{ac}\partial_b\chi^c + g_{bc}\partial_a\chi^c)\\ &= -\delta\chi^c\partial_c g_{ab} - g_{ac}\partial_b\delta\chi^c - g_{bc}\partial_a\delta\chi^c = -\mathcal{L}_{\delta\chi}g_{ab} \qquad (2.13)\\ \delta_{\chi,\lambda}(\delta\mathcal{A}) - \delta(\delta_{\chi,\lambda}\mathcal{A}) &= \iota_\chi\delta\mathcal{F} + \mathrm{d}(\iota_\chi\delta\mathcal{A} + \delta\lambda) - \delta(\iota_\chi\mathcal{F} + \mathrm{d}(\iota_\chi\mathcal{A} + \lambda))\\ &= \iota_{\delta\chi}\mathcal{F} + \mathrm{d}(\iota_{\delta\chi}\mathcal{A}) = -\mathcal{L}_{\delta\chi}\mathcal{A}. \qquad (2.14)\end{aligned}$$

Therefore, δ_{12} acts as an infinitesimal diffeomorphism along the vector field $-\delta\chi$. Hence we have

$$\theta[\delta_{12}] = -\,\mathrm{d}Q_{\delta\chi,0} - \iota_{\delta\chi}L. \tag{2.15}$$

[5] The Lagrangian density is Maxwell gauge invariant, so there is no contribution from λ to the left hand side.

Putting the three terms together, we obtain

$$\omega[\delta_{\chi,\lambda},\delta] = \mathrm{d}\left[\iota_\chi\theta[\delta] + \frac{1}{e^2}\delta\lambda * \mathcal{F} - \delta(Q_{\chi,\lambda}) + Q_{\delta\chi,0}\right] + \iota_\chi\,\mathrm{d}(\theta[\delta]) - \delta(\iota_\chi L) + \iota_{\delta\chi}L. \tag{2.16}$$

Using $\delta L = \mathrm{d}\theta$, we see that the latter terms on the right hand side cancel, and we just get an exact form. To get $\Omega[\delta_{\chi,\lambda},\delta]$, we just need to integrate this form over the Cauchy surface Σ. The result is a boundary integral given by

$$\Omega[\delta_{\chi,\lambda},\delta] = -\int_{\partial\Sigma}\Big(\delta(Q_{\chi,\lambda}) - Q_{\delta\chi,\delta\lambda} - \iota_\chi\theta[\delta]\Big). \tag{2.17}$$

Here we have combined $Q_{\delta\chi,0} + \frac{1}{e^2}\delta\lambda * \mathcal{F} = Q_{\delta\chi,\delta\lambda}$.

Because $Q_{\chi,\lambda}$ is linear in χ and λ, $\Omega[\delta_{\chi,\lambda},\delta]$ must be independent of $\delta\chi$ and $\delta\lambda$. We can therefore choose the behaviour of $\delta\chi$ and $\delta\lambda$ at $\partial\Sigma$ in any way we like, and this will not reduce the set of independent physically significant transformations under consideration. It will however have an effect on whether or not these transformations are integrable.

We now have a condition for whether a gauge transformation is large or not. Namely, it is large if and only if (2.17) is non-vanishing. Additionally, if χ is tangent to $\partial\Sigma$, then we can set $\delta\chi = \delta\lambda = 0$ and immediately obtain that $\int_{\partial\Sigma} Q_{\chi,\lambda}$ is the Hamiltonian charge generating the gauge transformation. The case where χ is not tangent to $\partial\Sigma$ requires a slightly more detailed analysis, and it is usually only possible to make such transformations integrable by making use of supplementary boundary conditions.

2.2.2 Charges of isolated systems

An *isolated* electromagnetic gravitational system is one for which $\mathcal{M}, g_{ab}$ is asymptotically flat and the field strength $\mathcal{F}$ falls off at some physically sensible rate in the asymptotic region. In systems of this type it is possible to choose for the Cauchy surface to have its boundary at $\mathscr{I}^+_-$ or $\mathscr{I}^-_+$, the past/future endpoints of future/past null infinity respectively. We will focus on these systems and make this choice in what follows.

The requirement that the systems we are analysing be isolated / asymptotically flat means that we will need to impose some gauge-invariant boundary conditions on the metric and gauge field at infinity. These are necessary for the specification of the theory.

We will also make some gauge choices. A full analysis would require that these gauge choices could always be reached by doing a small gauge transformation. If this were not the case, then the gauge choice would put a restriction on the allowed physical states in which the system could be. It is not too hard to show that the Maxwell gauge we will take is non-restrictive, but it is less obvious that the same is true of the coordinates we will pick. In fact, there is evidence to the contrary, e.g. [70]. Nevertheless, this gauge choice is almost always made in similar analyses, and we will do the same, leaving the resolution of this important issue for later work.

We will focus on the case $\partial\Sigma = \mathscr{I}^+_-$; the other choice $\partial\Sigma = \mathscr{I}^-_+$ proceeds in a very similar manner. We pick retarded Bondi coordinates (u, r, Θ^A), in which constant u surfaces are null, $g_{rA} = g_{rr} = 0$ and $\det(g_{AB}/r^2)$ is a function of Θ^A alone. We can write the metric near future null infinity (which is reached by taking $r \to \infty$) as [81, 114]

$$\begin{aligned} \mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &-\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}\Theta^A\,\mathrm{d}\Theta^B \\ &+ \frac{2m_b}{r}\,\mathrm{d}u^2 + rC_{AB}\,\mathrm{d}\Theta^A\,\mathrm{d}\Theta^B + D^BC_{AB}\,\mathrm{d}u\,\mathrm{d}\Theta^A \\ &+ \frac{1}{16r^2}C_{AB}C^{AB}\,\mathrm{d}u\,\mathrm{d}r \\ &+ \frac{1}{r}\left(\frac{4}{3}N_A + \frac{4}{3}u\partial_A m_b + \frac{1}{3}C_{AB}D_C C^{BC}\right)\mathrm{d}u\,\mathrm{d}\Theta^A \\ &+ \frac{1}{4}\gamma_{AB}C_{CD}C^{CD}\,\mathrm{d}\Theta^A\,\mathrm{d}\Theta^B \\ &+ \ldots \end{aligned} \tag{2.18}$$

The first line is the Minkowski metric. Later terms represent corrections to flat space. Constant u, r surfaces have spherical topology. C_{AB}, N_A, m_b all depend on u, Θ^A only, and capital Latin letters are lowered and raised with the unit round metric on the sphere γ_{AB} and its inverse γ^{AB}; its associated covariant derivative is D_A. C_{AB} is traceless with respect to γ_{AB}. The fields C_{AB}, N_A, m_b are related to each other, and to the Maxwell field, by the Einstein-Maxwell equations.

$\mathscr{I}^+_-$ is reached in these coordinates by considering a constant u, r surface, taking $r \to \infty$, and then taking $u \to -\infty$.

For the Maxwell field we choose retarded radial gauge $\mathcal{A}_r = 0$, $\mathcal{A}_u|_{\mathscr{I}^+} = 0$, and boundary conditions such that we can write near future null infinity

$$\mathcal{A} = \left(\frac{1}{r}E + O(r^{-2})\right)\mathrm{d}u + (A_A + O(r^{-1}))\,\mathrm{d}\Theta^A\,, \tag{2.19}$$

where E, A_A are functions of u, Θ^A only.

We are assuming that all physical states can be put into the forms above. As a consequence we need now only consider those gauge transformations which preserve them, which are sometimes referred to as 'residual' gauge transformations. The diffeomorphisms which preserve (2.18) are the Bondi-Metzner-Sachs (BMS) transformations [34, 130]. The components of a vector field ζ which generates a BMS transformation must take the following form at large r:

$$\begin{aligned}
\zeta^u &= Z \equiv f + \frac{1}{2}uD_AY^A, \\
\zeta^r &= -\frac{1}{2}rD_AY^A + \frac{1}{2}D^2Z - \frac{1}{4r}(C^{AB}D_AD_BZ + 2D_AC^{AB}D_BZ) + O(r^{-2}), \\
\zeta^A &= Y^A - \frac{1}{r}D^AZ + O(r^{-2}),
\end{aligned} \tag{2.20}$$

where f, Y^A depend only on Θ^A, and Y^A obeys the conformal Killing equation with respect to γ_{AB}, i.e. $D_AY_B + D_BY_A - \frac{1}{2}\gamma_{AB}D_CY^C = 0$. The function f is said to parametrise the 'supertranslation' part of ζ, and the vector Y^A the 'superrotation' part. A pure supertranslation is one with $Y^A = 0$, and a pure superrotation is one with $f = 0$. Note that ζ can only be exponentiated to a finite, non-singular diffeomorphism if Y is a global conformal Killing vector on the 2-sphere. Nevertheless, when considering infinitesimal transformations, it is valid to allow Y to take any value in the much larger space of general conformal Killing vectors.

This form for the vector field ζ is the one proposed in [23–25]. Before those papers, analysis of the BMS symmetries had focussed on the cases where Y is a global conformal Killing vector.

The action of this BMS transformation on $\mathcal{A}$ is given by

$$\mathcal{L}_\zeta\mathcal{A} = O(r^{-1})\,\mathrm{d}u + O(r^{-2})\,\mathrm{d}r + \mathrm{d}\Theta^A\left(Z\partial_uA_A + \partial_A(Y^BA_B) + O(r^{-1})\right). \tag{2.21}$$

The conditions from (2.19) on the u, Θ^A components of the gauge field are preserved by this transformation. However, the condition that $\mathcal{A}_r = 0$ is not. We will need to combine the BMS transformation with an appropriate Maxwell gauge transformation to preserve this condition. The allowed gauge transformations are given by $\mathcal{A} \to \mathcal{A} + \mathrm{d}\tau$, where

$$\tau = \varepsilon + \int \mathrm{d}r\,(\zeta^a\mathcal{F}_{ar} + \partial_r(\zeta^a\mathcal{A}_a)) = \varepsilon + O(r^{-1}). \tag{2.22}$$

ε is any function that depends only on Θ^A. It is a parameter for the Maxwell LGT.

In summary the remaining infinitesimal gauge transformations must have parameters of the above forms $\chi = \zeta$ and $\lambda = \tau$. We are now in a position to substitute our boundary conditions, gauge choices, and allowed residual gauge transformations into (2.17). A fair amount of algebra later, one obtains

$$\Omega[\delta_{\zeta,\tau}, \delta] = -\Big(\delta(H[f, Y, \varepsilon]) - H[\delta f, \delta Y, \delta\varepsilon] - T[f, Y]\Big), \tag{2.23}$$

where

$$H[f,Y,\varepsilon] = \int_{\mathscr{I}^+_-} \mathrm{d}^2\Theta\, \sqrt{\gamma} \left[\frac{m_b}{4\pi G} f + \left(\frac{N_A}{8\pi G} + \frac{E A_A}{e^2} \right) Y^A + \frac{E}{e^2}\varepsilon \right], \tag{2.24}$$

$$T[f,Y] = \int_{\mathscr{I}^+_-} \mathrm{d}^2\Theta\, \sqrt{\gamma} \left(f + \frac{1}{2} u D_A Y^A \right) \left(\frac{N_{BC}\delta C^{BC}}{16\pi G} + \frac{\partial_u A_A \delta A^A}{e^2} \right). \tag{2.25}$$

$N_{AB} = \partial_u C_{AB}$ is the 'Bondi news'.

The usual step now is to assume that we have boundary conditions such that $T[f,Y]$ vanishes; the standard choice is that the Bondi news N_{AB} and tangential components of the electric field $\partial_u A_A$ decay more quickly than $1/u$ as we approach $\mathscr{I}^+_-$. We can then set $\delta f = \delta Y^A = \delta\varepsilon = 0$, and find that supertranslations, superrotations and Maxwell large gauge transformations are all integrable, and are generated by $H[f,Y,\varepsilon]$.

The problem with this is that the boundary conditions at $\mathscr{I}^+_-$ are not preserved by all of these large gauge transformations. For example, one can calculate that a superrotation acts on the news as [81]

$$\delta N_{AB} = \mathcal{L}_Y N_{AB} - D_A D_B D_C Y^C + \frac{1}{2}\gamma_{AB} D^2 D_C Y^C. \tag{2.26}$$

This only preserves the condition given above on the news if Y^A is a global conformal Killing vector field on the round sphere, but we want to be able to include *all* superrotations, including those that are not global. Thus these charges and their resulting algebra will not be able to be exponentiated in a well-defined way.[6]

We will ignore this issue in this chapter, and therefore will assume that the large gauge transformation charge with supertranslation parameter f, superrotation parameter Y^A, and Maxwell LGT parameter ε of an asymptotically flat spacetime is given by $H[f,Y,\varepsilon]$. This is justified because, as far as the rest of this chapter is concerned, we are not actually concerned with the Hamiltonian interpretation of these charges. We will simply treat them as quantities which measure certain properties of the spacetime and its fields. From this point of view, the actions the charges generate are irrelevant.

[6] The fact that non-global superrotations will be singular at certain points on the sphere is a separate, and much less serious, obstruction to exponentiation of the algebra.

There are a few special cases of which we should make note. We define

$$M[f] = \int_{\mathscr{I}^+_-} \mathrm{d}^2\Theta \, \sqrt{\gamma} \frac{m_b}{4\pi G} f, \tag{2.27}$$

$$J[Y] = \int_{\mathscr{I}^+_-} \mathrm{d}^2\Theta \, \sqrt{\gamma} \left(\frac{N_A}{8\pi G} + \frac{E A_A}{e^2} \right) Y^A, \tag{2.28}$$

$$Q[\varepsilon] = \int_{\mathscr{I}^+_-} \mathrm{d}^2\Theta \, \sqrt{\gamma} \frac{E}{e^2} \varepsilon, \tag{2.29}$$

so that $H[f, Y, \varepsilon] = M[f] + J[Y] + Q[\varepsilon]$. $M = M[1]$ and $Q = Q[1]$ are the total mass and electric charge of the spacetime respectively. When f is an $l = 1$ spherical harmonic, $M[f]$ gives some component of the total momentum of the spacetime. When Y^A is a global conformal Killing vector field on the round sphere (these form a six-dimensional space), $J[Y]$ gives some component of the total angular momentum and boost charge of the spacetime. We will call $M[f]$ the mass weighted by f, $J[Y]$ the angular momentum weighted by Y^A, and $Q[\varepsilon]$ the electric charge weighted by ε. By substituting delta functions into the arguments of these three functions, we get

$$m(\Theta) = {}^2\epsilon \frac{m_b}{4\pi G}, \tag{2.30}$$

$$j_A(\Theta) = {}^2\epsilon \left(\frac{N_A}{8\pi G} + \frac{E A_A}{e^2} \right), \tag{2.31}$$

$$q(\Theta) = {}^2\epsilon \frac{E}{e^2}, \tag{2.32}$$

where ${}^2\epsilon$ is the pullback of $\sqrt{\gamma}\,\mathrm{d}^2\Theta$ to $\mathscr{I}^+_-$, and the right hand sides of these equations are each evaluated at the angle Θ on $\mathscr{I}^+_-$.

We should note that these are not the only soft charges one can construct. For example, by considering the electric-magnetic dual of Maxwell gauge transformations, one can construct soft magnetic charges [94]. However, these are the only ones which will be relevant in this chapter.

One possible interpretation of the above results is that there are independent physical gauge degrees of freedom associated to each null generator of $\mathscr{I}^+$. $m(\Theta), j_A(\Theta), q(\Theta)$ generate time translations, Lorentz transformations, and Maxwell gauge transformations on the null generator labelled by the angle Θ. It is for this reason that we will call m, j_A, q the angular densities of mass, angular momentum, and electric charge respectively.

2.3 Localisation of soft charges

The expressions found in Section 2.2 are all in terms of fields at infinity. However, if we want to discuss the soft charges of objects in the interior, we really want to be able to write down expressions in terms of fields near those objects. The objective of this section is to carry out this procedure of localisation. In particular, let $\tilde{\Sigma}$ be a surface such that $\mathscr{I}^+_-$ is only one component of $\partial\tilde{\Sigma}$, and define $S = \partial\tilde{\Sigma} \setminus \mathscr{I}^+_-$. We will write down the soft charges in terms of integrals over S.

2.3.1 Maxwell LGT charge

We initially focus on the soft electric charge, which can be written as

$$Q[\varepsilon] = \frac{1}{e^2} \int_{\mathscr{I}^+_-} \varepsilon * \mathcal{F}. \tag{2.33}$$

By the Maxwell equations of motion $\mathrm{d} * \mathcal{F} = 0$, we have

$$Q^{\tilde{\Sigma}}[\varepsilon] = \frac{1}{e^2} \int_{\tilde{\Sigma}} \mathrm{d}\varepsilon \wedge *\mathcal{F} = Q[\varepsilon] - \frac{1}{e^2} \int_S \varepsilon * \mathcal{F}. \tag{2.34}$$

Suppose that $Q^{\tilde{\Sigma}}[\varepsilon]$ vanishes. Then clearly $Q[\varepsilon] = \frac{1}{e^2} \int_S \varepsilon * \mathcal{F}$ is an expression for the soft charge associated to ε. This equality holds for all solutions of the Maxwell equations of motion, but more importantly the expressions for $Q[\varepsilon]$ as defined at $\mathscr{I}^+_-$ and as defined at S are completely physically equivalent in a Hamiltonian sense, in that they generate the same flow on phase space. In this way we have localised the charge $Q[\varepsilon]$ to S.

The simplest example is obtained by setting $\varepsilon = 1$. $Q = Q[\varepsilon] = \frac{1}{e^2} \int_{\mathscr{I}^+_-} *\mathcal{F}$ is then just the total electric charge of the spacetime. $\mathrm{d}\varepsilon$ vanishes, so $Q^{\tilde{\Sigma}}[\varepsilon] = 0$, and we find an equally valid expression for the total electric charge, $Q = Q^S = \frac{1}{e^2} \int_S *\mathcal{F}$.

We want to repeat this exercise for a more general choice of ε. In fact a similar kind of scenario will continue to arise during this chapter. We will now lay out some machinery for application to the general case, before specialising to the electromagnetic LGT charges, and then other examples in later sections.

Suppose we have an integral of the form $I[f] = \int_{\mathscr{I}^+_-} f\beta$, where f is a weight function on $\mathscr{I}^+_-$, and β is a closed 2-form. Let $I^{\tilde{\Sigma}}[F] = \int_{\partial\tilde{\Sigma}} F\beta = \int_{\tilde{\Sigma}} \mathrm{d}F \wedge \beta$, where $F = f$ on $\mathscr{I}^+_-$. A sufficient condition for $I^{\tilde{\Sigma}}[F] = 0$ is the vanishing of the pullback of $\mathrm{d}F \wedge \beta$ to $\tilde{\Sigma}$. This can be written as

$$n_a(*\beta)^{ab}\partial_b F = 0, \tag{2.35}$$

where n is a non-vanishing normal to $\tilde{\Sigma}$. In other words, F need only be constant along integral curves of $n_a(*\beta)^{ab}$. If we choose $n = \mathrm{d}t$ where t is a level-surface function specifying $\tilde{\Sigma}$, this vector field is divergence-free.[7] Hence its integral curves can only end at $\partial\tilde{\Sigma}$. Therefore the map

[7] In the case where $\beta = *\mathcal{F}$, this is just Gauss' law.

U_β which takes each point in $\partial\tilde{\Sigma}$ to the other end of the integral curve through that point is a well-defined involution of $\partial\tilde{\Sigma}$.[8] We will make the assumption that $U_\beta(\mathscr{I}^+_-) \cap \mathscr{I}^+_-$ is empty, i.e. that each integral curve intersects $\mathscr{I}^+_-$ no more than once. We then have $S_\beta \equiv U_\beta(\mathscr{I}^+_-) \subset S$. Picking F to be constant along integral curves, we can write

$$I[f] = \int_{\mathscr{I}^+_-} f\beta = \int_{S_\beta} (f \circ U_\beta)\beta = I^S[f]. \tag{2.36}$$

We will call the right hand side of the above equation the 'localised' form of $I[f]$. Let $\beta|_{\mathscr{I}^+_-}, \beta|_S$ be the pullbacks of β to $\mathscr{I}^+_-$ and S respectively. Since f is arbitrary in the above equation, this is really a relation between these two forms:

$$\beta|_{\mathscr{I}^+_-} = U^*_\beta \beta|_S. \tag{2.37}$$

Note that the well-defined-ness of the right hand side is contingent on the smoothness of U_β, and this property is not guaranteed. We will ignore this issue. The right hand side is certainly well-defined where U_β is smooth; we will treat it as a formal expression wherever this does not hold.

So consider now the soft electric charges. In the absence of matter $*\mathcal{F}$ is closed by the Maxwell equations. In this case the vector field along which the weight function should be constant is just the electric field $E^a = n_b \mathcal{F}^{ab}$. Assuming that $U_{*\mathcal{F}}(\mathscr{I}^+_-) \cap \mathscr{I}^+_- = \emptyset$, we can thus write down a localised form of the LGT charge

$$Q^S[\varepsilon] = \frac{1}{e^2} \int_{S_{*\mathcal{F}}} (\varepsilon \circ U_{*\mathcal{F}}) * \mathcal{F}, \tag{2.38}$$

or, in terms of angular charge densities $q = \frac{1}{e^2} * \mathcal{F}|_{\mathscr{I}^+_-}$ and $q_S = \frac{1}{e^2} * \mathcal{F}|_S$,

$$q = U^*_{*\mathcal{F}} q_S. \tag{2.39}$$

This expression successfully localises an arbitrary soft electric charge to a finite subregion of spacetime. This localisation has a simple geometric interpretation – the Maxwell gauge transformation parameter must be constant along electric field lines. This serves to demonstrate that electric field lines have an important role to play in the story of soft electric charges.

An example is provided in Figure 2.2. There are a few noteworthy features of this example. For example, there are regions of S that are completely inaccessible to the localisation procedure.

[8] If the vector field vanishes at a point in $\partial\tilde{\Sigma}$, then there *is* no integral curve through that point. We will ignore this issue. It is not really a problem, because at the points where the vector field vanishes there will be no contribution to $I[f]$ anyway.

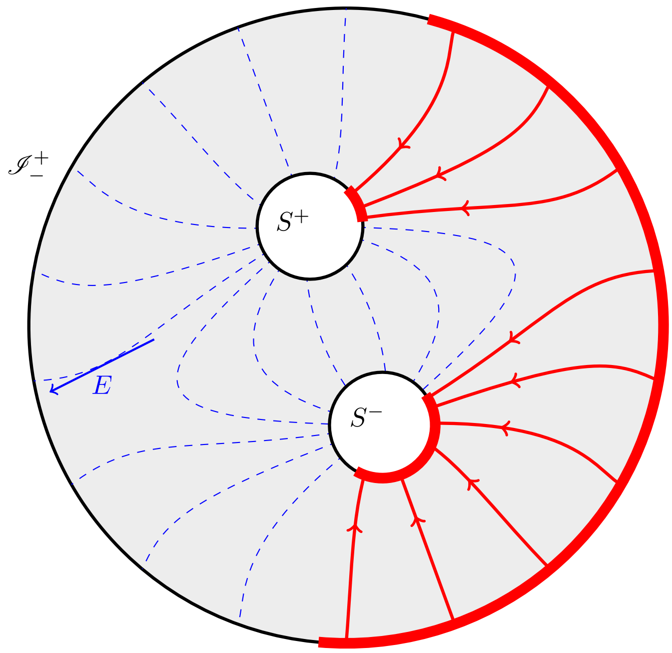


Figure 2.2: Example of localisation of soft electric charge by propagation along electric field lines. This is a top-down view of the surface $\tilde{\Sigma}$. The outer boundary denotes $\mathscr{I}^+_-$, and the inner boundaries S^+ and S^- comprise $S = \partial\tilde{\Sigma} \setminus \mathscr{I}^+_-$. The lines in the interior are the integral curves of the electric field E. Consider a soft charge with asymptotic support in the thick red region of $\mathscr{I}^+_-$. Once localised, this will take the form of an integral with support in the thick red regions of S^+ and S^-.

These are those for which the appropriate electric field line both starts and ends on S. Also, the localisation map $U_{*\mathcal{F}}$ can be seen to be non-smooth at certain points, where its image jumps between S^+ and S^-. However, note that at these points the electric field changes direction, and the charge density vanishes. Hence we do not need a localisation at these points anyway. It seems likely that a similar kind of thing happens in the generic case.

2.3.2 Gauge independence

In this procedure, the choice of surface $\tilde{\Sigma}$ interpolating between infinity and the interior surface S played a key role, and we should address to what extent the map U_β depends on this choice. In diffeomorphism-independent theories, the choice of $\tilde{\Sigma}$ is a choice of gauge. So we need to be careful that the procedure we are discussing is not gauge-dependent, as this would call into question the physical significance of the following results.

We should note straight away that what we are doing *is* gauge-independent. To see this, suppose $\tilde{\Sigma}_1, \tilde{\Sigma}_2$ are two different surfaces sharing a boundary $S \cup \mathscr{I}^+_-$, and suppose f is a weight function on $\mathscr{I}^+_-$. By the procedure we described, one can extend f to two functions $F_{1,2}$ on the two surfaces $\tilde{\Sigma}_1, \tilde{\Sigma}_2$ respectively, constant on field lines. Then we have

$$\int_S F_1\beta = \underbrace{\int_{\tilde{\Sigma}_1} \mathrm{d}F_1 \wedge \beta}_{=0} + \int_{\mathscr{I}^+_-} F_1\beta = \int_{\mathscr{I}^+_-} f\beta \tag{2.40}$$

and

$$\int_S F_2\beta = \underbrace{\int_{\tilde{\Sigma}_2} \mathrm{d}F_2 \wedge \beta}_{=0} + \int_{\mathscr{I}^+_-} F_2\beta = \int_{\mathscr{I}^+_-} f\beta, \tag{2.41}$$

so

$$\int_S F_1\beta = \int_S F_2\beta. \tag{2.42}$$

Thus, the results of localising on $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are exactly the same, so the procedure cannot depend on this choice.

On the other hand, the map U_β itself *does* depend on the choice of $\tilde{\Sigma}$. This can be seen by considering the two field lines on the two surfaces $\tilde{\Sigma}_{1,2}$ which start at the same point $x \in \mathscr{I}^+_-$. There is no reason that these field lines have to intersect S at the same point, as the 2-form β can be specified fairly arbitrarily near the two surfaces.

This may appear at first to be a problem, but there is no contradiction here. The gauge-dependence of U_β means the integrands in (2.42) are not unique. However, we are only interested in the values of integrals over S. Even if the integrands change, the integrals can (and in this case must) remain the same.

The key point we wish to make in this subsection is that although the map U_β itself is gauge-dependent, the formula (2.37) is not. All of the general results we will obtain in the rest of the chapter are derived from this formula, and so are physically meaningful.

We should just view the map U_β, and the involvement of field lines, as an intuitive geometric tool for understanding what the procedure involves, but we should always be aware that it is not gauge-invariant by itself. In practical cases where we are actually trying to carry out localisation for specific field configurations, this gauge-dependence could actually end up being useful. We could use it to pick a convenient surface $\tilde{\Sigma}$ on which the localisation map U_β takes an easy-to-use form. This would be analogous to the choice of convenient coordinates in complicated spacetimes. In that case, the coordinates are not gauge-independent, but one must still use them to compute many gauge-invariant quantities.

2.3.3 BMS charge

We now wish to carry out the same procedure for the BMS charges $M[f]$ and $J[Y^A]$. In order to do so we will make some quite restrictive assumptions about the spacetime we are dealing with. It seems likely that a construction can be found that does not make these assumptions. However, the assumptions hold in the main topic of interest (the study of stationary black holes), so we will use them in what follows.

First, suppose χ is a Killing vector of g_{ab}, and λ is such that $\mathrm{d}\lambda = -\mathcal{L}_\chi \mathcal{A}$. We refer to such a χ, λ as a Killing pair. Then we have $\theta[\delta_{\chi,\lambda}] = 0$, since θ is linear in the field variations, and these just vanish. Using the formula $\theta[\delta_{\chi,\lambda}] = \mathrm{d}Q_{\chi,\lambda} + \iota_\chi L$, and the fact that the Ricci scalar R vanishes whenever the equations of motion hold, we therefore have

$$\begin{aligned} 0 &= \mathrm{d}Q_{\chi,\lambda} + \frac{1}{2e^2}\iota_\chi(\mathcal{F} \wedge *\mathcal{F}) \\ &= \mathrm{d}\left[Q_{\chi,\lambda} - \frac{1}{2e^2}((\iota_\chi \mathcal{A} + \lambda) * \mathcal{F} - \mathcal{A} \wedge (\iota_\chi * \mathcal{F}))\right], \end{aligned} \tag{2.43}$$

where the second line is most easily reached by repeated application of the magic formula $\mathcal{L}_\chi \alpha = \iota_\chi \mathrm{d}\alpha + \mathrm{d}(\iota_\chi \alpha)$ and substitution of Maxwell's equation $\mathrm{d} * \mathcal{F} = 0$. Hence we discover that for each Killing pair χ, λ we have an associated exact 2-form given by the contents of the square brackets, explicitly $\mathrm{d}N[\chi, \lambda] = 0$ where

$$N[\chi, \lambda] = \frac{1}{16\pi G} * \mathrm{d}\chi + \frac{1}{2e^2}((\iota_\chi \mathcal{A} + \lambda) * \mathcal{F} + \mathcal{A} \wedge (\iota_\chi * \mathcal{F})). \tag{2.44}$$

In the case $\chi = 0, \lambda =$ constant, this reduces to Maxwell's equation. Note that this result is a special case of the generalised Noether theorem obtained in [22].

The assumption we make to localise the supertranslation charge $M[f]$ is that spacetime is stationary with timelike Killing vector field $k = \partial_u$, with both the metric and Maxwell field invariant under the action of $\mathcal{L}_k$. It can then be shown that

$$N[k,0]|_{\mathscr{I}^+_-} = \epsilon \frac{m_b}{8\pi G}. \tag{2.45}$$

Therefore, the supertranslation charge in the stationary case can be written as

$$M[f] = 2\int_{\mathscr{I}^+_-} f N[k,0]. \tag{2.46}$$

Similarly, to localise the superrotation charge $J[Y]$, we assume that spacetime is axially symmetric with rotational Killing vector field ψ, and that we can write ψ as a BMS transformation generating vector field of the form (2.20), with $f = 0$ and $Y^A = \psi^A$ a global conformal Killing vector field on the sphere. We then find that

$$N[\psi,0]|_{\mathscr{I}^+_-} = \epsilon\left(\frac{N_A}{8\pi G} + \frac{EA_A}{e^2}\right)\psi^A. \tag{2.47}$$

So, in the case that we can write $Y^A = h\psi^A$ for some function h, we can write the superrotation charge in the axially symmetric case as

$$J[h\psi] = \int_{\mathscr{I}^+_-} hN[\psi,0]. \tag{2.48}$$

To localise these charges, we can now just follow the same procedure as for the electric charge. We find

$$M[f] = M^S[f] = 2\int_{S_{N[t,0]}} (f \circ U_{N[t,0]})N[t,0] \tag{2.49}$$

and

$$J[h\psi] = J^S[h\psi] = \int_{S_{N[\psi,0]}} (h \circ U_{N[\psi,0]})N[\psi,0]. \tag{2.50}$$

In terms of angular densities, we have

$$m = U^*_{N[t,0]} m_S, \tag{2.51}$$

$$j^A\psi_A = U^*_{N[\psi,0]} j_S, \tag{2.52}$$

where m_S, j_S are defined as the pullbacks of $2N[t,0], N[\psi,0]$ to S respectively.

2.4 Thermodynamics of soft hair

In [80, 81], it was pointed out that in the context of the new soft charges, the argument for the black hole information paradox may be flawed. Any black hole spacetime may be mapped to a physically different black hole spacetime by the action of a large gauge transformation or large diffeomorphism. Using a stationary black hole as the background and applying a spontaneous symmetry breaking argument, one observe that quantum black holes obtain a set of Goldstone modes. These are referred to as soft hair, and they invalidate the no-hair theorem in the quantum context. The authors of [80] conjecture that the soft hair will be sufficient to restore the information that is seemingly lost in black hole evaporation. Whether this is true is still a matter of debate, and we will not attempt to settle it here. For some viewpoints, see [37, 120, 139].

Nevertheless, there are still problems one can hope to solve in this context without running into too much controversy. A natural question to ask is whether one can obtain versions of the laws of black hole mechanics which respect the soft charges, and whether one can give these a thermodynamical interpretation. We will refer to these as the laws of black hole mechanics at every angle, and their derivation and exposition is the objective of this section.

It is worth noting that an appropriate zeroth law and second law have already been shown to hold at every angle. The conventional zeroth law is the statement that the surface gravity of a stationary black hole is constant over the horizon. This trivially implies that the surface gravity is pointwise constant, which is the zeroth law at every angle. The conventional second law is the statement that the area of the event horizon of the black hole cannot decrease. A second law at every angle would then have to be that the expansion of each null generator of the horizon is non-negative. But showing that such a statement holds is a step in most proofs of the traditional second law. See for example [79, Lemma 9.2.2].

The third law is much less concrete than the other three. One way of stating it is: it is impossible for the surface gravity of an initially non-extremal (i.e. non-vanishing surface gravity) black hole to be reduced to zero everywhere on the horizon in a finite number of steps. It seems natural to guess that the generalisation to every angle should be one of two possibilities: it is impossible for the surface gravity of a black hole to be reduced from a non-zero value to zero at either a single point on the horizon, or in the neighbourhood of any point on the horizon, in a finite number of steps. Since the traditional third law has not been rigorously proven, we will not attempt to carry out a proof of a third law at every angle at this stage. We will only comment that if one can be shown to be true, then it seems likely that the other can too.

It remains to generalise the first law of black hole mechanics to one concerning charges at every angle. The traditional first law describes a relation that must hold if we perturb a black hole by a small amount. In Einstein-Maxwell theory, this is

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q, \tag{2.53}$$

where M is the mass of the black hole, Ω its angular velocity, J its angular momentum, Φ its electric potential, Q its electric charge, κ its surface gravity, and A its area. It is reasonable to expect that one can find a similar identity that relates instead these quantities at every angle. In a sense, the above law is an integral one: it relates quantities that are obtained by integrating over a time slice of the event horizon. The first law at every angle that we obtain is in this sense a differential one, relating quantities that are defined pointwise on the event horizon. Schematically, it takes the form

$$\delta m = \frac{\kappa}{8\pi G}\delta a + \nabla\cdot l + \Omega\delta j + \Phi\delta q, \tag{2.54}$$

where m, a, j, q are densities that integrate to their capitalised counterparts, and $\nabla\cdot l$ is the divergence of a vector field l tangential to the horizon that depends linearly on the field perturbations. As we will discuss, l has a natural thermodynamical interpretation as a heat flow tangential to the horizon. Note that upon integration over the horizon this divegence term disappears, and we obtain again the conventional first law.

In this section we will focus on asymptotically flat stationary spacetimes containing a single non-extremal electrically charged black hole. In these spacetimes we have access to a stationary Killing vector field k and a rotational Killing vector field ψ. We will normalise these such that k has unit norm and the orbits of ψ have period 2π at infinity. As above, we will write the electrostatic potential of the black hole relative to infinity as Φ, and its angular velocity as Ω. The vector field $\xi = k - \Omega\psi$ is the Killing vector field that generates the event horizon.

2.4.1 Smarr's formula

Before obtaining the first law, we will warm up with a generalised version of Smarr's formula [135]. Consider the conserved 2-form $N \equiv N[\xi, -\Phi]$, and let $\tilde{\Sigma}$ be a surface with boundary given by the disjoint union of $\mathscr{I}^+_-$ and $\mathcal{S}$, the bifurcate 2-surface where the past event horizon $\mathcal{H}^-$ and future event horizon $\mathcal{H}^+$ meet. Figure 2.3 depicts this scenario.

At $\mathscr{I}^+_-$, N pulls back to

$$N|_{\mathscr{I}^+_-} = \frac{1}{2}m - \Omega\psi^A j_A - \frac{1}{2}\Phi q, \tag{2.55}$$

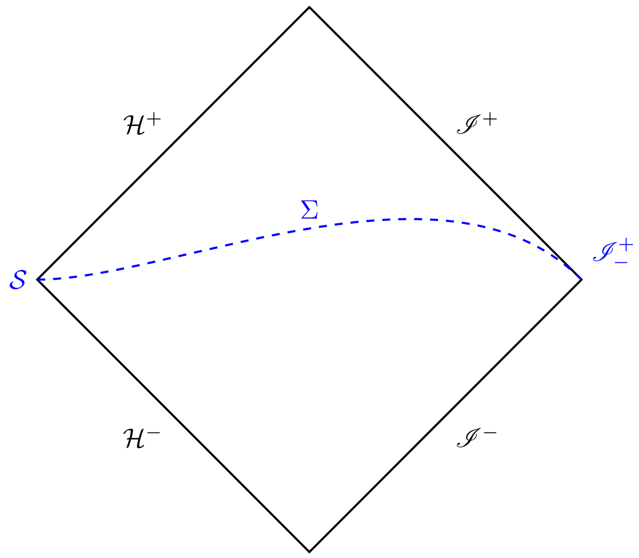


Figure 2.3: The domain of dependence of the surface $\tilde{\Sigma}$ in a stationary black hole spacetime chosen such that $\partial\tilde{\Sigma} = \mathscr{I}^+_- \cup \mathcal{S}$.

and at $\mathcal{S}$ it pulls back to

$$N|_{\mathcal{S}} = \frac{\kappa}{8\pi G} a, \tag{2.56}$$

where κ is the surface gravity of the event horizon, and a is the induced area element on the horizon. Hence, integrating N over $\partial\tilde{\Sigma}$, one finds

$$\frac{1}{2}M - \Omega J - \frac{1}{2}\Phi Q = \frac{\kappa}{8\pi G} A, \tag{2.57}$$

where M is the mass of the black hole, J is its angular momentum, Q is its electric charge, and A is its area. This is the Smarr formula.

(2.57) is an equation that only applies to the angular zero modes of the large gauge and large diffeomorphism charges. One can generalise it to one that has an angular dependence by using the localisation method established previously. Let z be some function on the sphere. Then we have

$$\frac{1}{2}M[z] - \Omega J[z\psi] - \frac{1}{2}\Phi Q[z] = \int_{\mathscr{I}^+_-} zN = \int_{\mathcal{S}_N} (z \circ U_N) N. \tag{2.58}$$

Using (2.56), and defining the weighted black hole area

$$A[z] = \int_{\mathcal{S}_N} (z \circ U_N) a, \tag{2.59}$$

one obtains

$$\frac{1}{2}M[z] - \Omega J[z\psi] - \frac{1}{2}\Phi Q[z] = \frac{\kappa}{8\pi G} A[z]. \tag{2.60}$$

This is an angular generalisation of the Smarr formula. One can also express it in terms of charge densities if we set z to a delta function. We then obtain

$$\frac{1}{2}m - \Omega\psi^A j_A - \frac{1}{2}\Phi q = \frac{\kappa}{8\pi G}U_N^* a. \tag{2.61}$$

2.4.2 The first law

To obtain the first law, we will need a conserved 2-form that depends linearly on field variations. To find one, we consider now (2.17), replicated below for convenience, when χ, λ is a Killing pair (they only need to be a Killing pair for the background, not the varied fields).

$$\Omega[\delta_{\chi,\lambda}, \delta] = -\int_{\partial\Sigma} \Big(\delta(Q_{\chi,\lambda}) - Q_{\delta\chi,\delta\lambda} - \iota_\chi\theta[\delta]\Big).$$

The left hand side clearly vanishes, since the presymplectic structure is linear in the variations. Thus in this case the integrand on the right hand side is a closed form. Explicitly, $\mathrm{d}T[\chi,\lambda] = 0$, where

$$T[\chi,\lambda] = \delta(Q_{\chi,\lambda}) - Q_{\delta\chi,\delta\lambda} - \iota_\chi\theta[\delta]. \tag{2.62}$$

So again consider a stationary charged black hole spacetime, and in particular let us choose the Killing pair $\chi, \lambda = \xi, -\Phi$ as in the previous section. Let $\tilde{\Sigma}$ be as in the previous section, and define $T \equiv T_{\xi,\Phi}$. At $\mathscr{I}^+_-$, T pulls back to

$$T|_{\mathscr{I}^+_-} = \delta m - \Omega\psi^A \delta j_A - \Phi\delta q \tag{2.63}$$

and at $\mathcal{S}$ it pulls back to

$$T|_{\mathcal{S}} = \frac{\kappa}{8\pi G}\delta a \tag{2.64}$$

Therefore if we integrate $\mathrm{d}T$ over $\tilde{\Sigma}$, we obtain the celebrated first law of black hole mechanics in its standard form,

$$\delta M = \Omega\delta J + \Phi\delta Q + \frac{\kappa}{8\pi G}\delta A. \tag{2.65}$$

To find an angle-dependent first law, we can proceed in the usual way, but the vector field $n_a(*T)^{ab}$ is a little difficult to deal with. Our solution is to start by splitting up T.

Primes will denote varied fields:

$$g'_{ab} = g_{ab} + \delta g_{ab}, \qquad \mathcal{A}' = \mathcal{A} + \delta\mathcal{A}. \tag{2.66}$$

More generally primes will denote quantities derived using the primed fields. One can write $T = M - N$, where N is defined as before, and

$$M = N' + \frac{1}{2e^2}\Big(\iota_\chi(\mathcal{A}' \wedge (*\mathcal{F})' - \mathcal{A} \wedge *\mathcal{F}) + \lambda((*\mathcal{F})' - *\mathcal{F})\Big)$$
$$- \iota_\chi\Big(\frac{1}{16\pi G}\epsilon_a \nabla_b (g'^{ab} - g^{ab} g_{cd} g'^{cd}) + \frac{1}{e^2}(\mathcal{A}' - \mathcal{A}) \wedge *\mathcal{F}\Big) \quad (2.67)$$

We have shown previously that $\mathrm{d}N = 0$. We also have that $\mathrm{d}M = \mathrm{d}T + \mathrm{d}N = 0$. Therefore, each of these forms is individually conserved. Because T is linear in the field variations, M and N are infinitesimally close to each other. Furthermore M and N generically have non-zero parts that do not depend on the field variations. Therefore, M and N are much larger than their difference. This will become useful in what follows.

The left hand side of the generalised first law will take the form $\int_{\mathscr{I}^+_-} zT$, where z is some function. Define two functions v and w with the property that $v = w = z$ at $\mathscr{I}^+_-$. Then we have

$$\int_{\mathscr{I}^+_-} zT = \int_{\mathscr{I}^+_-} vM - \int_{\mathscr{I}^+_-} wN. \quad (2.68)$$

Now we will localise each integral on the right hand side individually. We get

$$\int_{\mathscr{I}^+_-} zT = \int_{\mathcal{S}_M} (v \circ U_M)M - \int_{\mathcal{S}_N} (w \circ U_N)N. \quad (2.69)$$

Using a delta function for z we get an expression in terms of densities

$$T = U_M^* M - U_N^* N = U_N^*\big((U_M \circ U_N^{-1})^* M - N\big) \quad (2.70)$$

Note that $U_M \circ U_N^{-1}$ is a diffeomorphism of $\mathcal{S}$. Since the difference between N and M is much smaller than either, it is safe to make the assumption that this diffeomorphism is infinitesimally close to the identity. Let it be characterised by a vector field $\hat{l}$ tangent to $\mathcal{S}$. $\hat{l}$ depends linearly on the field variations $\delta g_{ab}, \delta\mathcal{A}$, but *not* locally. We then have

$$U_M^* M - U_N^* N = U_N^*(M + \mathcal{L}_{\hat{l}} M - N) = U_N^*(T + \mathrm{d}(\iota_{\hat{l}} N)). \quad (2.71)$$

But N, T at $\mathcal{S}$ are just given by $\frac{\kappa}{8\pi G} a$ and $\frac{\kappa}{8\pi G}\delta a$ respectively. Therefore we obtain

$$\delta m - \Omega \psi^A \delta j_A - \Phi \delta q = \frac{\kappa}{8\pi G} U_N^* \left[\delta a + \mathrm{d}(\iota_{\hat{l}} a)\right]. \quad (2.72)$$

One can expand the right hand side to find

$$\delta m - \Omega \psi^A \delta j_A - \Phi \delta q = \frac{\kappa}{8\pi G}\delta(U_N^* a) + \frac{\kappa}{8\pi G} U_N^* \, \mathrm{d}(\iota_l a), \quad (2.73)$$

where l is the vector field on the horizon generating the diffeomorphism $U_{N+P}U_N^{-1}$, and

$$P = M - N' = \frac{1}{2e^2}\delta(\iota_\chi(\mathcal{A}\wedge *\mathcal{F}) + \lambda *\mathcal{F}) - \iota_\chi\left(\frac{1}{16\pi G}\epsilon_a(\nabla_b \delta g'^{ab} - \nabla^a \delta g) + \frac{1}{e^2}\delta\mathcal{A}\wedge *\mathcal{F}\right). \quad (2.74)$$

Equation (2.73) is the first law of black hole mechanics at every angle. The right hand side contains a variation of the horizon area density, but also a term that appears to correspond to a horizon surface current l. This horizon surface current could equally well be interpreted as a surface current on the celestial sphere, by pulling it back through the map U_N.

Finally, note that if we integrate (2.73) over $\mathscr{I}^+_-$ against a weight function f, we get this generalisation of the first law in integral form:

$$\delta M[f] - \Omega\delta J[f\psi] - \Phi\delta Q[f] = \frac{\kappa}{8\pi G}\delta A[f] + \frac{\kappa}{8\pi G}\int_{\mathcal{S}_N} l(f\circ U_N)a. \quad (2.75)$$

Note that if we set $f = 1$, the rightmost term vanishes, and we just get back the first law in its usual form. The conventional first law is thus just one of the infinity of first laws provided by the above expression.

2.5 Discussion

In this chapter we have proposed a simple method for the localisation of soft charges to the interior of a spacetime. We have also obtained a set of laws governing the soft charges of an asymptotically flat spacetime containing a black hole. The first three are:

0. *The surface gravity of a stationary black hole has vanishing gradient.*
1. *A perturbation to a stationary black hole obeys* (2.73).
2. *The expansion along each null generator of the horizon is non-negative.*

The third law we conjecture to have two possible forms:

3. *It is impossible to reduce the surface gravity at any point (strong) / in the neighbourhood of any point (weak) on the horizon from a positive value to zero in a finite number of steps.*[9]

The original four laws of black hole mechanics are widely believed to arise from the thermodynamics of the microscopic physics of a near equilibrium black hole. The Bekenstein-Hawking

[9] 'A finite number of steps' is quite a nebulous phrase. It could mean the application of a finite number of basic discrete unitary operators. Alternatively, it could mean a finite amount of continuous time evolution. As is unfortunately quite common when one deals with the third law, the precise meaning of this phrase is not entirely clear.

entropy $S = A/4G$ strongly suggests that the microscopic states are in some way distributed over the black hole horizon. Therefore it seems reasonable to hope that this generalisation of the laws of black hole mechanics, which applies to each point on the horizon individually, has the potential to shed some new light on the microscopic degrees of freedom, which in this context are the soft hairs at each angle.

The above laws suggest a natural generalisation of black hole temperature and entropy, that should be expected to hold near equilibrium. Let x be a point in the horizon. We propose that the entropy density $s(x)$ and temperature $t(x)$ of the black hole at x should be given by

$$s(x) = \frac{a(x)}{4G}, \qquad t(x) = \frac{\kappa(x)}{2\pi}. \tag{2.76}$$

The non-negative expansion of the horizon implies that this definition of entropy density obeys the second law of thermodynamics. It would be of interest to compare the angular Hawking spectrum of a near equilibrium black hole temperature with the above value.

We have written $\kappa(x)$ in (2.76), but in the equilibrium case, κ does not depend on x. In the non-equilibrium case, there is no Killing vector which generates the black hole horizon, and there is no widely accepted definition of surface gravity (although there have been attempts at such a definition [83, 108]). Clearly, in order for the proposal (2.76) to make sense, one needs such a definition. We leave exploration of this to future work.

Equations (2.76) could have been guessed without the above analysis, but the rightmost term in (2.73) suggests another, less obvious, part to this analogy. We propose that the 1-form

$$\iota_{l(x)} \frac{a(x)}{4G} \tag{2.77}$$

provides a natural candidate for the heat current of the horizon at the point x for an approximately stationary black hole. This describes how energy is exchanged between the microscopic degrees of freedom of the black hole (i.e. the soft hair), and so it should hopefully provide some insight on how these are coupled together. The heat current is derived directly from the Wald-Noether charge density N and the presymplectic potential form θ. These are both intimately related to the information content of the spacetime, and this makes this definition particularly appealing. Of interest is the fact that the heat current appears to be constructed in a non-local manner from fields outside of the black hole. This follows from the appearance in its definition of maps that propagate along the integral curves of certain vector fields, and perhaps reflects the non-local behaviour that any quantum theory of gravity is believed to exhibit.

It should be noted that the notion of an entropy density really only makes sense when the microscopic local degrees of freedom are weakly correlated with each other. This is a reflection of the fact that entropy is subadditive:

$$S_A + S_B \geq S_{AB}. \tag{2.78}$$

Here, we have split some system into two parts A and B, and S_A, S_B, S_{AB} are the entropies of A, B and their union respectively. We only have approximate additivity

$$S_A + S_B \approx S_{AB} \tag{2.79}$$

when correlations between A and B are negligible. The existence of an entropy density which one can integrate to give the total entropy of a system is just the continuous version of (2.79), and so requires weak correlations. In the case where the microscopic degrees of freedom are highly correlated, we would need to include non-local contribution in any formula for the total entropy.

With this in mind, our proposal for the entropy density seems to be at odds with the suggestion that the degrees of freedom in a black hole are very quickly scrambled, and thus highly correlated [82, 110, 132, 143]. We see two possiblities for evading this tension. The first uses the fact that the results we have obtained apply to a small perturbation to a stationary black hole. After such a perturbation has been made, there is a small period of time before it becomes scrambled. At this point, so long as the initial perturbation involves weak correlations, the concept of an entropy density for the perturbations makes sense. So perhaps it is the case that our results only apply to these perturbations in this window of time.

The other possibility is that the entropy density we have suggested actually *is* a non-local quantity, despite appearances. This is because of the key role played by the map U_N, which must be used to relate the entropy density at a point x to the energy density and other physical quantities at x. Perhaps this non-locality is sufficient to account for the highly correlated state of the black hole.

We do not claim that either of these two ideas is absolutely correct, and the true way to resolve the tension may be completely different. Regardless, the fact that we have a density which integrates to the total entropy, and which obeys a form of the laws of thermodynamics, makes us feel safe to at least tentatively continue referring to it as the entropy density.

Besides the rather open-ended goal of exploring the consequences of the thermodynamical interpretation of the above laws, there are many directions in which this work could be taken in the future. We list a few below.

The maps that propagate along integral curves played a key role in the localisation of soft charges. This suggests that if one formulates theories with these charges in such a way that the integral curves are given an explicit role, then it may be possible to obtain some new insights. This would be interesting to explore.

It would be of use to understand the connection (if any) between the present work and the study of bit threads [69]. Both invoke ideas of divergence-free vector fields, and 1-to-1 maps between degrees of freedom and the flow lines of these vector fields. One possible test of this would be to use the localisation formulas we describe to understand how the flux of soft charges depend on subregions at the boundary. This could be compared to the properties of bit threads.

The localisation of the gravitational soft charges described in this chapter only works for some soft charges, and only in spacetimes which permit Killing fields. One should try to generalise the method so that Killing fields are not required.

The soft charges at infinity form a closed algebra when they are combined with Poisson brackets. The localised soft charges also form an algebra in the same way. The localisation procedure in this chapter provides a relation between these two algebras. One might expect that this relationship is an isomorphism, and it would be worthwhile to verify this.

The angular momentum term in the first law is $\Omega\psi^A\delta j_A$. It only has something to say about a certain component of the angular momentum density, namely the component in the direction of the rotational Killing field ψ. It might be worth exploring whether the other component of the angular momentum density has a role to play in black hole thermodynamics.

It would be useful to provide concrete examples of the applicability of the generalised first law described in this chapter. For example, it is known that throwing an asymmetric configuration of matter into a stationary black hole results in a change in its soft hairdo [81]. The configuration of the spacetime after this process is a perturbation of the initial configuration, and therefore obeys the first law. This should be checked. In a similar vein, numerical simulations of perturbed black holes should obey the first law, and this might be worth testing.

Finally, one should evaluate the right hand side of the first law in the case of the Kerr-Newman black hole. The resulting explicit expression for l may contain some interesting information. We have not yet managed to obtain an answer to this calculation which isn't prohibitively algebraically complicated. This could possibly be simplified by a clever choice of localising surface $\tilde{\Sigma}$, but we have not found such a choice.

Chapter 3

Unambiguous Phase Spaces for Subregions

3.1 Introduction

Recall the recipe for the symplectic structure of a subregion, described in Section 1.1. First, there is a procedure for deriving a certain differential form ω from the Lagrangian density. Then, one picks a partial Cauchy surface Σ. Finally, one integrates ω over Σ to obtain the symplectic structure

$$\Omega = \int_\Sigma \omega. \tag{3.1}$$

Unfortunately, this recipe suffers from a significant ambiguity which we also described in Section 1.1. The form ω is only defined up to the addition of a certain class of exact forms. Under such a change $\omega \to \omega + \mathrm{d}\beta$, the symplectic structure changes by a boundary integral, $\Omega \to \Omega + \int_{\partial\Sigma} \beta$. If Σ has no boundary, then Ω is unmodified. But in many cases of physical significance Σ does have a boundary (which may be either finite or asymptotic), and the ambiguity is a cause for genuine concern. Without a completely well-defined symplectic structure, the theory itself is ill-defined.

Several approaches to dealing with this ambiguity have arisen. One might note that the ambiguity only affects physics at the boundary. Thus, if one is only concerned with physics deep in the interior of the subregion, one might argue that the ambiguity is irrelevant, so one may simply ignore it. However, in gauge theories this is untenable, due to the presence of non-local degrees of freedom which lead to correlations between the physics near the boundary and in the interior. Even if there is no gauge symmetry, this point of view is spoiled by the fact that very often we *are* concerned with physics at the boundary. In fact in many cases the physics at the boundary is the main subject of interest.

One example where the boundary physics is important arises in the case of a black hole spacetime, where it is natural to choose Σ such that $\partial\Sigma$ intersects the event horizon – with this choice, one is studying the physics on one side of the black hole. One may show that certain quantities, such as the Wald entropy, are unaffected by the ambiguity in the stationary case[1] [97], but there are non-trivial consequences of a more complicated nature. For example, in [71], the charge algebra of large gauge transformations at black hole horizons was studied. These charges are highly sensitive to the ambiguity. The authors of that paper make a particular choice of boundary term, simply to make the large gauge transformations that they were interested in integrable (in the Hamiltonian sense). They (deliberately) provide no a priori justification for this prescription.

Common to these approaches is the implicit belief that the recipe for the symplectic structure is completely correct, and that the boundary ambiguity must be fixed by additional, situation-dependent, considerations.

In this chapter, we will take an alternate viewpoint. We will argue that the boundary ambiguity is not actually present, and that it only arises because the recipe is incomplete.

We will take a point of view in which we assume the classical field theory arises as the classical limit of some quantum theory. We find this a useful approach because it provides an intuitive way to think about the physical meaning of the Poisson bracket. By the Dirac relation

$$\{A, B\} = \lim_{\hbar \to 0} \frac{1}{i\hbar} \langle [\hat{A}, \hat{B}] \rangle \,, \tag{3.2}$$

the Poisson bracket of two observables A and B should really be thought of as the leading order contribution to the expectation value of the commutator of two corresponding operators $\hat{A}$ and $\hat{B}$, in the classical limit. Using saddlepoint approximations in a path integral allows us to compute the effects of operators in the classical field configuration, and this provides a convenient way to frame the derivation.

However, this quantum approach is not strictly necessary for the results we obtain. When we use a leading order saddlepoint approximation, we are really just evaluating the classical equations of motion (sometimes in the presence of sources). When we consider the order of operators, in the classical limit this just corresponds to inserting sources at different times. One could rewrite the whole derivation purely in terms of these classical concepts, and everything would follow through. This would essentially just amount to a change of notation. Again, we

[1] In the non-stationary case, there is still an important ambiguity in the black hole entropy [57, 96, 156].

only use the quantum context because we find that it leads to a clearer intuitive picture for the meaning of the Poisson bracket.

Thus, the resolution of the ambiguity we describe should not be thought of as coming from the quantum theory. Indeed, there is no reason that a classical theory involving a symplectic form with a different boundary term to the one we propose should not be quantisable.

The resolution instead comes from the following. When considering a subregion of space, we do not want there to be anything special about the boundary of that subregion, such as strange additional boundary conditions. Moreover, the subregion should consistently and smoothly embed inside of the larger spacetime, so that, for example, an excitation can travel from the inside of the subregion to the outside without experiencing anything unusual. It is this principle which will dictate to us the correct boundary term.

By a direct derivation from the Lagrangian path integral, we will show that the symplectic structure associated to Σ is in fact given by a contour integral of ω around Σ. To be more precise, let $\mathcal{U}$ be any open submanifold of spacetime containing Σ. Then we find that the symplectic structure is given by

$$\Omega = \int_{\partial\mathcal{U}} \omega. \tag{3.3}$$

Because $\partial(\partial\mathcal{U}) = \emptyset$, the ambiguity $\omega \to \omega + \mathrm{d}\beta$ is no longer an issue, as it does not result in a change in Ω as defined by (3.3). One may recover an expression resembling (3.1) by taking the limit as $\mathcal{U}$ shrinks to contain only Σ, and by using certain causality conditions. The covariant phase space itself is also slightly modified in our approach. Whereas before it was given by the space of solutions to the equations of motion, we argue that it should instead be given by a space of field configurations which obey the equations of motion everywhere *except* at Σ.

The outline of our derivation is as follows. First we define observables in the region associated with Σ as observables which depend only on the field configuration on Σ. Then we compute the expectation value of the commutator of two such observables by inserting them into the path integral. Using the fundamental relation that arises in the classical limit between the commutator and the classical Poisson bracket, this allows us to obtain a Poisson structure for Σ. Finally, we invert this Poisson structure to obtain the symplectic structure.

Before proceeding, we should say that there have been previous attempts to address the ambiguity in the symplectic form. However, in our opinion these attempts are insufficient to address the case we are considering, for various reasons. For example, [75] resolved the ambiguity when Σ is a complete Cauchy surface, making use of boundary conditions at $\partial\Sigma$, but in our case Σ is a partial Cauchy surface, and there are no such boundary conditions. Another attempt is in [100], but it seems to us that paper is simply describing a way of singling out a particular boundary term, without giving a physical reason why it is the correct one. Additionally, it only applies to first order theories, and we want to avoid such a restriction.

We believe our result does not suffer from these issues. Our main argument is given in Section 3.2. We then describe the application of our result to a simple example in Section 3.3. We conclude with some remarks and speculation in Section 3.4 on the consequences of our results for gauge symmetry, edge modes, and entanglement.

3.2 Disambiguation of the covariant phase space

There is a method for recovering the correct symplectic structure from the semiclassical path integral. The outline is that one may compute the expectation value of the commutator of two observables by inserting the appropriate combination of corresponding operators into the path integral, and then taking the limit as the time-separation of these operators goes to zero. Using the relation between the quantum commutator and the classical Poisson bracket allows one to obtain a Poisson structure for the theory, which may then be inverted to obtain the symplectic structure.

In this section we will employ this method, but restrict to observables which are accessible from within a subregion. We will assume that there is a sensible operator interpretation for the subregion observables. This will allow us to obtain a well-defined and unambiguous symplectic structure for that subregion.

Our results will apply to a broad class of field theories with gauge symmetries. However, strictly speaking we should restrict to non-gravitational theories, i.e. those without diffeomorphism invariance. The restriction is necessary because a theory with diffeomorphism invariance does not have any local observables, and so the notion of the degrees of freedom in a subregion becomes much more subtle. In particular, if we are not careful, diffeomorphisms may move excitations in or out of the subregion under consideration.

Nevertheless, we expect that it is possible to extend our analysis to theories of gravity, if one defines subregions in the correct way. In particular, subregions should be defined in a gauge-invariant manner (e.g. the exterior of the event horizon is certainly a gauge-invariant region of spacetime). Then we expect that similar results will apply. This would be interesting to verify, and is possibly connected to work in [42, 62, 63, 136].

3.2.1 Observables in a subregion

A subregion is defined as a partial Cauchy surface $\Sigma \subset \mathcal{M}$. An observable $W[\phi]$ is a gauge-invariant function[2] on configuration space $\mathcal{C}$ that only depends on $\phi(x)$, and its derivatives normal to Σ, if $x \in \Sigma$. In the classical theory, for any given field configuration ϕ, $W[\phi]$ is just a number which may be directly computed. The analogue of this number in the quantum theory is the expectation value $\langle \hat{W} \rangle$ of some operator $\hat{W}$ associated with W. The classical limit is well-defined only if $\langle \hat{W} \rangle$ converges to the classical number W as $\hbar \to 0$.[3]

We will now recall how this works for the semiclassical path integral, which for a quantum field theory with action S is given by

$$\mathcal{Z} = \int \mathrm{D}\phi \exp(iS/\hbar). \tag{3.4}$$

The integration is done over all field configurations which obey the boundary conditions. In the semiclassical limit $\hbar \to 0$, a saddlepoint approximation reveals that the path integral is dominated by configurations for which S is extremised, i.e. those for which $\delta S = 0$ for an arbitrary choice of $\delta\phi$, or equivalently for which the equations of motion $E = 0$ are obeyed. We assume that the boundary conditions are chosen such that there is only one solution to the equations of motion (up to gauge symmetry), which we will denote ϕ_0. Then the path integral may be approximated by

$$\mathcal{Z} = \exp(iS[\phi_0]/\hbar), \tag{3.5}$$

up to further factors which represent the contribution of quantum fluctuations away from ϕ_0. Truncating these factors amounts to restricting to tree-level Feynman diagrams. This should be a

[2] It would be interesting to expand this analysis to include gauge-dependent observables by following ideas in [118]. We leave this to future work.

[3] Throughout this chapter, we are simply assuming that we have been given some pre-existing pairing $W, \hat{W}$ of classical and quantum operators. The only condition that we need this pairing to obey is $\langle \hat{W} \rangle \to W$ as $\hbar \to 0$. In general, given a classical observable W, there can be many corresponding quantum operators $\hat{W}$, or none. We will simply assume that there is at least one $\hat{W}$ for each classical observable W we wish to consider – the final result for the unambiguous Ω will not depend on which one. Indeed, as we described in the introduction, everything we do could be rephrased purely in terms of the classical theory, so Ω should not upon the details of the quantisation. We should also clarify what $\langle \hat{W} \rangle$ means. For each classical configuration ϕ, there is in the quantum theory a 'coherent state' $|\phi\rangle$ which closely approximates ϕ. Then $W[\phi]$ is equal to $\langle \hat{W} \rangle_\phi = \langle \phi | \hat{W} | \phi \rangle$. This is also what $W[\phi]$ means when we insert it in a path integral, if we assume that the path integral is derived in terms of coherent states. Again, the Ω we derive should not depend on the fine details of the construction of the coherent states – only that they approximate the classical states in the right way.

good approximation whenever perturbation theory works, i.e. at weak coupling. We will assume that for the theory we are considering this truncation is a valid approximation, but it may be useful in the future to investigate the higher loop corrections in the following derivation.[4] It is not entirely clear whether our results extend to the case of strong coupling, but it may be possible to explore that regime by analytic continuation of the coupling constants.

The expectation value $\langle \hat{W} \rangle$ is defined by

$$\langle \hat{W} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}\phi \, W \exp(iS/\hbar). \tag{3.6}$$

We may compute this expectation value by using a sourced path integral. First we introduce a sourced action $S(\epsilon) = S + \epsilon W$. The sourced path integral is then defined by

$$\mathcal{Z}(\epsilon) = \int \mathrm{D}\phi \exp(iS(\epsilon)/\hbar), \tag{3.7}$$

and the range of this integral is the same as for (3.4). Then we clearly have

$$\langle \hat{W} \rangle = -i\hbar \frac{\partial}{\partial \epsilon} \log \mathcal{Z}(\epsilon)\bigg|_{\epsilon=0}. \tag{3.8}$$

In order to evaluate this expression, we need to know the value of $\mathcal{Z}(\epsilon)$ for small ϵ, and this can be done by again using a saddlepoint approximation. The sourced path integral is dominated by configurations for which the sourced action $S(\epsilon)$ is extremised. For such configurations we have

$$\delta(S + \epsilon W) = \int_{\mathcal{M}} \delta\phi \cdot E + \epsilon \delta W = 0 \tag{3.9}$$

for all possible choices of $\delta\phi$.

Suppose that ϕ_W is a field configuration obeying (3.9) and the boundary conditions. We assume that ϕ_W is unique, up to gauge symmetry. In the case that there is gauge symmetry, we will just pick one ϕ_W out of the possible gauge equivalent configurations – all the following statements will be invariant with respect to this choice. We further assume that ϕ_W is a smooth function of ϵ satisfying $\phi_W|_{\epsilon=0} = \phi_0$. Then ϕ_W defines a smooth path with parameter ϵ in the space of field configurations. Let $\delta_W\phi$ denote its tangent vector at ϕ_0.

Expanding in powers of ϵ, the value of the sourced action at $\phi = \phi_W$ is given by

$$S[\phi_W] + \epsilon W[\phi_W] = S[\phi_0] + \epsilon \int_{\mathcal{M}} \delta_W\phi \cdot \underbrace{E[\phi_0]}_{=0} + \epsilon W[\phi_0] + \mathcal{O}\Big(\epsilon^2\Big). \tag{3.10}$$

[4] Another possibility here would be to carry out a coherent state decomposition of the path integral. These are states which have a good classical limit, and so permit an approximation of the type in (3.5). Corrections to (3.5) arise from the overlap between different coherent states, and this overlap also permits an interpretation in the classical limit. Such an approach would perhaps bear some similarity with the formalism in [28].

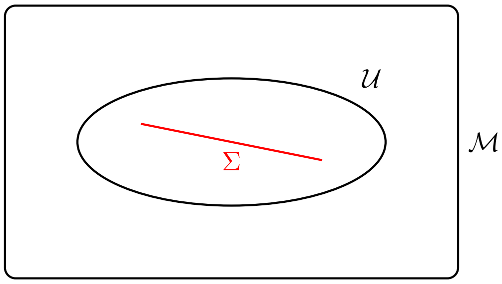


Figure 3.1: $\Sigma \subset \mathcal{U} \subset \mathcal{M}$. The action is deformed by a source term at the codimension one surface Σ.

Using now the saddlepoint approximation

$$\mathcal{Z}(\epsilon) = \exp(iS[\phi_W]/\hbar) \tag{3.11}$$

and (3.8), we find

$$\langle \hat{W} \rangle = W[\phi_0]. \tag{3.12}$$

This expression is valid up to subleading in $\hbar$ corrections. Such corrections are negligible in the $\hbar \to 0$ limit. Therefore, given our assumptions above, the expectation value of W attains its classical value in the classical limit, which is well-defined.

Before moving on to the next subsection, it will be useful to derive some further results concerning $\delta_W \phi$ and $S[\phi_W]$. Let $\mathcal{U}$ be a D-dimensional open submanifold of $\mathcal{M}$ such that $\Sigma \subset \mathcal{U}$, and consider first the case where $\delta\phi$ vanishes in $\mathcal{U}$. Then we have $\delta W = 0$, and may therefore write

$$\left[\int_{\mathcal{M} \setminus \mathcal{U}} \delta\phi \cdot E \right]_{\phi_W} = 0. \tag{3.13}$$

By the arbitrarity of $\delta\phi$, we may conclude that the equations of motion $E = 0$ are obeyed outside of $\mathcal{U}$. Assuming this is true, (3.9) therefore reduces to

$$\left[\int_{\mathcal{U}} \delta\phi \cdot E + \epsilon \delta W \right]_{\phi_W} = 0. \tag{3.14}$$

One may write

$$E[\phi_W] = E[\phi_0 + \epsilon \delta_W \phi] = \underbrace{E[\phi_0]}_{=0} + \epsilon \mathcal{S}(\delta_W \phi) + \mathcal{O}\left(\epsilon^2\right), \tag{3.15}$$

where $\mathcal{S}$ is a linear differential operator characterising the linearised equations of motion. For example, in the case of a scalar field described by $E = \Box\phi = 0$, we have $\mathcal{S} = \Box$. Substituting this into (3.14), and considering only the $\mathcal{O}(\epsilon)$ term, one finds

$$\delta W = -\int_{\mathcal{U}} \delta\phi \cdot \mathcal{S}(\delta_W \phi), \tag{3.16}$$

where this equation is understood to hold at $\phi = \phi_0$.

Next, we will make the connection with the formalism described in Section 1.1. One may substitute $\delta L = \mathrm{d}\theta + \delta\phi \cdot E$ into (3.14) to obtain

$$\left[\int_{\mathcal{U}} \delta L - \int_{\partial\mathcal{U}} \theta[\delta\phi] + \epsilon\delta W\right]_{\phi_W} = 0. \tag{3.17}$$

δL and θ may be viewed as 1-forms in configuration space, since they depend linearly on $\delta\phi$. By considering configuration space Lie derivatives with respect to $\delta_W\phi$, we can obtain expansions about ϕ_0 of these objects, in powers of ϵ. One finds

$$[\delta L]_{\phi_W} = \left[\delta L + \epsilon\delta_W\delta L + \mathcal{O}\left(\epsilon^2\right)\right]_{\phi_0}, \tag{3.18}$$

and

$$[\theta[\delta\phi]]_{\phi_W} = \left[\theta[\delta\phi] + \epsilon(\delta_W(\theta[\delta\phi]) - \theta[[\delta_W\phi, \delta\phi]]) + \mathcal{O}\left(\epsilon^2\right)\right]_{\phi_0}, \tag{3.19}$$

where $[\delta_W\phi, \delta\phi]$ is the configuration space commutator of the two vectors $\delta_W\phi, \delta\phi$. Substituting these into (3.17), and considering only the $\mathcal{O}(\epsilon)$ term, one obtains

$$\delta W = \int_{\partial\mathcal{U}} (\delta_W(\theta[\delta\phi]) - \theta[[\delta_W\phi, \delta\phi]]) - \int_{\mathcal{U}} \delta_W\delta L, \tag{3.20}$$

which holds at $\phi = \phi_0$. Noting that

$$\delta_W\delta L = \delta\delta_W L = \delta\Big(\delta_W\phi \cdot E + \mathrm{d}(\theta[\delta_W\phi])\Big) \tag{3.21}$$

$$= \delta_W\phi \cdot \mathcal{S}(\delta\phi) + \mathrm{d}(\delta(\theta[\delta_W\phi]))\,, \tag{3.22}$$

we may write (3.20) as

$$\delta W = \int_{\partial\mathcal{U}} (\delta_W(\theta[\delta\phi]) - \delta(\theta[\delta_W\phi]) - \theta[[\delta_W\phi, \delta\phi]]) - \int_{\mathcal{U}} \delta_W\phi \cdot \mathcal{S}(\delta\phi). \tag{3.23}$$

Referring back to (1.18), one recognises the first integrand as $\omega[\delta_W\phi, \delta\phi]$. Therefore, using (3.16) to eliminate δW, we find

$$\int_{\partial\mathcal{U}} \omega[\delta_W\phi, \delta\phi] = \int_{\mathcal{U}} \Big(\delta_W\phi \cdot \mathcal{S}(\delta\phi) - \delta\phi \cdot \mathcal{S}(\delta_W\phi)\Big). \tag{3.24}$$

Finally, we will obtain an expression for the $\mathcal{O}(\epsilon^2)$ term in the value of the sourced action at $\phi = \phi_W$. We have

$$S[\phi_W] + \epsilon W[\phi_W] = S[\phi_0] + \frac{1}{2}\epsilon^2\left[\delta_W\left(\int_{\mathcal{M}} \delta_W\phi \cdot E\right)\right]_{\phi_0} + \epsilon\Big(W[\phi_0] + \epsilon\delta_W W[\phi_0]\Big) + \mathcal{O}\Big(\epsilon^3\Big) \tag{3.25}$$

$$= S[\phi_0] + \epsilon W[\phi_0] + \epsilon^2\left[\frac{1}{2}\int_{\mathcal{M}} \delta_W\phi \cdot \mathcal{S}(\delta_W\phi) + \delta_W W\right]_{\phi_0} + \mathcal{O}\Big(\epsilon^3\Big) \tag{3.26}$$

$$= S[\phi_0] + \epsilon W[\phi_0] - \frac{1}{2}\epsilon^2\left[\int_{\mathcal{U}} \delta_W\phi \cdot \mathcal{S}(\delta_W\phi)\right]_{\phi_0} + \mathcal{O}\Big(\epsilon^3\Big). \tag{3.27}$$

In the last line, we used (3.13) to restrict the integral to $\mathcal{U}$, and then used (3.16) with $\delta\phi = \delta_W\phi$ to eliminate $\delta_W W$.

3.2.2 Operator composition and the Poisson bracket

We need a Poisson bracket for observables on Σ. Such a bracket should agree with the commutator in the classical limit. To be precise, for any two observables A, B on Σ, we require

$$\frac{1}{i\hbar}\langle[\hat{A}, \hat{B}]\rangle \to \{A, B\} \quad \text{as} \quad \hbar \to 0. \tag{3.28}$$

This can be taken as the definition of the Poisson bracket.

In order for the commutator to make sense, we need a notion of operator ordering. In the path integral, this is implemented by 'causal' boundary conditions, i.e. those such that $\delta_W\phi$ only has support in $J^+(\Sigma)$, for any observable W on Σ. Here $J^+(\Sigma)$ is the causal future of Σ, i.e. the set of points in $\mathcal{M}$ which can be reached by following a future-directed[5] causal curve starting in Σ.

We will assume that our boundary conditions are causal. Operator ordering then translates directly to time ordering. An insertion of $[A, B]$ into the path integral really means an insertion of the combination $A(t)B(-t) - B(t)A(-t)$, where $A(t), B(t)$ are versions of A, B which have been displaced a certain amount in time t. One takes the limit $t \to 0$ from above, after having carried out the path integration[6].

[5] One must assume that $\mathcal{M}$ has a time-orientation.

[6] It is important that this limit takes place after the path integration. If one were to take the limit first, one would find a vanishing commutator, since A, B are only c-numbers in the path integral, so

$$\lim_{t\to 0}\Big(A(t)B(-t) - B(t)A(-t)\Big) = AB - BA = 0.$$

When doing the path integral first, an $\mathcal{O}(1/t)$ number of paths will have significant contributions, so a non-zero quantity will result from the $t \to 0$ limit.

Clearly we will need a notion of time-displacement for the observables A, B on Σ. To that end, let $\Sigma(t) \subset \mathcal{M}$ be a smooth 1-parameter family of partial Cauchy surfaces such that $\Sigma(0) = \Sigma$, and such that

$$\Sigma(t_1) \subset J^+(\Sigma(t_2)) \quad \text{if} \quad t_1 > t_2. \tag{3.29}$$

This condition says that $\Sigma(t_1)$ is to the future of $\Sigma(t_2)$ whenever $t_1 > t_2$. Now let $A(t), B(t)$ be a pair of observables on each $\Sigma(t)$, smooth in the parameter t, and such that $A(0) = A$, $B(0) = B$. $A(t), B(t)$ are the time-displaced observables.

It is not clear that the way in which this time-displacement should be chosen to happen is unique, i.e. that there is a unique choice of the surfaces $\Sigma(t)$ and the time-displaced observables $A(t), B(t)$. The choice is clearly not completely free, as there are a number of consistency conditions that must be obeyed for the operator interpretation to make sense, one of which we will take advantage of below. Nevertheless, the final expression we will obtain can be conveniently written in notation that is independent of this choice. The actual impact of this choice on the value of the symplectic structure appears to be as a kind of regularisation, and we will describe an example of this in Section 3.3.

By the above considerations, we may write the expectation value of AB as

$$\langle \hat{A}\hat{B} \rangle = \lim_{t \to 0} \int \mathrm{D}\phi \, A(t) B(-t) \exp(iS/\hbar). \tag{3.30}$$

As in the previous subsection, we can compute this expectation value using a sourced path integral. First let

$$S(\sigma, \tau) = S + \sigma A(t) + \tau B(-t) \tag{3.31}$$

be the sourced action, and define the sourced path integral as

$$\mathcal{Z}(\sigma, \tau) = \int \mathrm{D}\phi \exp(iS(\sigma, \tau)/\hbar). \tag{3.32}$$

Then we have

$$\langle \hat{A}\hat{B} \rangle = -\hbar^2 \lim_{t \to 0} \left[\frac{1}{\mathcal{Z}(\sigma, \tau)} \frac{\partial^2 \mathcal{Z}(\sigma, \tau)}{\partial \sigma \partial \tau} \right]_{\sigma = \tau = 0} \tag{3.33}$$

We will again use a saddlepoint approximation for this computation, writing

$$\mathcal{Z}(\sigma, \tau) = \exp(iS_{\text{extremal}}(\sigma, \tau)/\hbar), \tag{3.34}$$

where $S_{\text{extremal}}(\sigma, \tau)$ is the extremal value of the sourced action. The most efficient way to compute this quantity is to use the results of the previous subsection, with the substitutions

$$\epsilon W \to \sigma A(t) + \tau B(-t), \tag{3.35}$$

$$\epsilon \delta_W \phi \to \sigma \delta_{A(t)} \phi + \tau \delta_{B(-t)} \phi. \tag{3.36}$$

When one does this, it is important to ensure that $\mathcal{U}$ is chosen to contain both $\Sigma(t)$ and $\Sigma(-t)$. Using these substitutions in (3.27), one finds

$$S_{\text{extremal}}(\sigma,\tau) = S + \sigma A(t) + \tau B(-t) - \frac{1}{2}\int_{\mathcal{U}} \Big(\sigma\delta_{A(t)}\phi + \tau\delta_{B(-t)}\phi\Big)\cdot\mathcal{S}\Big(\sigma\delta_{A(t)}\phi + \tau\delta_{B(-t)}\phi\Big) + \dots \tag{3.37}$$

where the right-hand side should be evaluated at $\phi = \phi_0$, and the ellipsis contains terms of cubic order and higher in σ,τ. At this point, it is simple to apply (3.33), and one obtains

$$\langle\hat{A}\hat{B}\rangle = \lim_{t\to 0}\left(A(t)B(-t) + \frac{i\hbar}{2}\int_{\mathcal{U}}\left[\delta_{A(t)}\phi\cdot\mathcal{S}(\delta_{B(-t)}\phi) + \delta_{B(-t)}\phi\cdot\mathcal{S}(\delta_{A(t)}\phi)\right]\right) + \mathcal{O}\Big(\hbar^2\Big). \tag{3.38}$$

We have $\lim_{t\to 0} A(t)B(-t) = AB$. Also, by our assumptions about causality, an operator insertion at t cannot affect an observation at $-t$, so we have

$$\int_{\mathcal{U}} \delta_{A(t)}\phi\cdot\mathcal{S}(\delta_{B(-t)}\phi) = -\delta_{A(t)}B(-t) = 0. \tag{3.39}$$

Thus, we may write

$$\langle\hat{A}\hat{B}\rangle = AB + \frac{i\hbar}{2}\lim_{t\to 0}\int_{\mathcal{U}}\delta_{B(-t)}\phi\cdot\mathcal{S}(\delta_{A(t)}\phi) + \mathcal{O}\Big(\hbar^2\Big). \tag{3.40}$$

A useful result arises from the following consistency condition:

$$\langle\hat{W}\rangle^* = \langle\hat{W}^\dagger\rangle\,, \tag{3.41}$$

i.e. the complex conjugate of the expectation value is the expectation value of the Hermitian conjugate. We will assume that A,B are both real observables, which means that their corresponding quantum operators are Hermitian. Therefore,

$$\langle\hat{A}\hat{B}\rangle^* = \langle\hat{B}^\dagger\hat{A}^\dagger\rangle = \langle\hat{B}\hat{A}\rangle\,. \tag{3.42}$$

Taking the complex conjugate of (3.40), we have

$$\langle\hat{A}\hat{B}\rangle^* = AB - \frac{i\hbar}{2}\lim_{t\to 0}\int_{\mathcal{U}}\delta_{B(-t)}\phi\cdot\mathcal{S}(\delta_{A(t)}\phi) + \mathcal{O}\Big(\hbar^2\Big). \tag{3.43}$$

Also, swapping A and B in (3.40) yields

$$\langle\hat{B}\hat{A}\rangle = AB + \frac{i\hbar}{2}\lim_{t\to 0}\int_{\mathcal{U}}\delta_{A(-t)}\phi\cdot\mathcal{S}(\delta_{B(t)}\phi) + \mathcal{O}\Big(\hbar^2\Big). \tag{3.44}$$

By (3.42), the right-hand sides of the above two equations are equal. As a consequence we find

$$\lim_{t\to 0}\Big(\delta_{A(-t)}B(t) + \delta_{B(-t)}A(t)\Big) = -\lim_{t\to 0}\int_{\mathcal{U}}\left[\delta_{A(-t)}\phi\cdot\mathcal{S}(\delta_{B(t)}\phi) + \delta_{B(-t)}\phi\cdot\mathcal{S}(\delta_{A(t)}\phi)\right] = \mathcal{O}(\hbar). \tag{3.45}$$

Now let us evaluate the commutator. From (3.40), we have

$$\langle[\hat{A},\hat{B}]\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\hat{A}\rangle \tag{3.46}$$
$$= \frac{i\hbar}{2}\lim_{t\to 0}\int_{\mathcal{U}}[\delta_{B(-t)}\phi\cdot\mathcal{S}(\delta_{A(t)}\phi) - \delta_{A(-t)}\phi\cdot\mathcal{S}(\delta_{B(t)}\phi)] + \mathcal{O}\left(\hbar^2\right) \tag{3.47}$$
$$= -i\hbar\lim_{t\to 0}\int_{\mathcal{U}}\delta_{A(-t)}\phi\cdot\mathcal{S}(\delta_{B(t)}\phi) + \mathcal{O}\left(\hbar^2\right), \tag{3.48}$$

where we used (3.45) to reach the third line. By the defining relation for the Poisson bracket (3.28), we may therefore take the $\hbar\to 0$ limit to obtain

$$\{A,B\} = -\lim_{t\to 0}\int_{\mathcal{U}}\delta_{A(-t)}\phi\cdot\mathcal{S}(\delta_{B(t)}\phi) = \lim_{t\to 0}\delta_{A(-t)}B(t) = -\lim_{t\to 0}\delta_{B(-t)}A(t). \tag{3.49}$$

There is one more useful way in which we may write the Poisson bracket. Using our causality assumptions, we have

$$\{A,B\} = \{A,B\} - \lim_{t\to 0}\delta_{B(t)}A(-t) = \lim_{t\to 0}\left(\delta_{A(-t)}B(t) - \delta_{B(-t)}A(t)\right). \tag{3.50}$$

We see that this Poisson bracket is essentially the same as the Peierls bracket [55, 56, 124] restricted to observables on Σ. We may further use (3.16) to obtain

$$\{A,B\} = \lim_{t\to 0}\int_{\mathcal{U}}[\delta_{B(t)}\phi\cdot\mathcal{S}(\delta_{A(-t)}\phi) - \delta_{A(-t)}\phi\cdot\mathcal{S}(\delta_{B(t)}\phi)]. \tag{3.51}$$

The right-hand side we recognise from (3.24), which we may therefore use to write

$$\{A,B\} = -\lim_{t\to 0}\int_{\partial\mathcal{U}}\omega[\delta_{A(-t)}\phi,\delta_{B(t)}\phi]. \tag{3.52}$$

3.2.3 Phase space and symplectic structure

Let us write $\{A,B\} = \Pi(A,B)$. Π is an antisymmetric bivector on configuration space known as the Poisson structure. We may view Π as a map $A\mapsto\Pi(A)$ from observables on Σ to vector fields on configuration space, defined by $\Pi(A)(B) = \Pi(A,B)$. $\Pi(A)$ is known as the Hamiltonian vector field associated to the observable A, and may be thought of as the field variation resulting from the application of A as an operator. From (3.49) it is clear that

$$\Pi(A) = \lim_{t\to 0}\delta_{A(-t)}\phi. \tag{3.53}$$

So this Poisson structure is in agreement with the saddlepoint approximation.

A standard result in Poisson geometry says that the commutator of any two Hamiltonian vector fields gives a third. By Frobenius' theorem, the Hamiltonian vector fields therefore span the tangent spaces to the leaves[7] of a regular foliation of the configuration space. These leaves are known as symplectic leaves. Let $\mathcal{P}_\Sigma$ be the symplectic leaf containing ϕ_0.

Suppose one starts at ϕ_0 and applies some operators at Σ.[8] This corresponds to flowing along some combination of Hamiltonian vector fields. During this flow, the state must remain in $\mathcal{P}_\Sigma$, because the Hamiltonian vector fields are tangent to $\mathcal{P}_\Sigma$. Conversely, it is possible to reach any field configuration in $\mathcal{P}_\Sigma$ by flowing along the appropriate Hamiltonian vector fields, i.e. by applying the right operators at Σ.

So $\mathcal{P}_\Sigma$ is the space of field configurations which can be explored by the application of operators at Σ. It therefore makes sense to use $\mathcal{P}_\Sigma$ as the phase space for the degrees of freedom on Σ.

It remains to obtain a symplectic structure on $\mathcal{P}_\Sigma$. This is the inverse of $-\Pi$ restricted to $\mathcal{P}_\Sigma$, which another standard result in Poisson geometry says is unique. Note that

$$\delta A = -\int_{\mathcal{U}} \delta\phi \cdot \mathcal{S}(\Pi(A)). \tag{3.54}$$

Therefore, the map

$$\Omega(\hat{\delta}\phi) = \int_{\mathcal{U}} \delta\phi \cdot \mathcal{S}(\hat{\delta}\phi) \tag{3.55}$$

inverts $-\Pi$. Restricting this to $\mathcal{P}_\Sigma$, we have

$$\Omega(\hat{\delta}\phi) = \lim_{t\to 0} \int_{\mathcal{U}} \delta\phi(-t) \cdot \mathcal{S}(\hat{\delta}\phi(t)). \tag{3.56}$$

In this expression, $\delta\phi(-t), \delta\phi(t)$ are field variations originating from operator insertions on $\Sigma(-t)$ and $\Sigma(t)$ respectively. Using causality, this may be written

$$\Omega(\hat{\delta}\phi) = \lim_{t\to 0} \int_{\mathcal{U}} [\delta\phi(t) \cdot \mathcal{S}(\hat{\delta}\phi(-t)) - \hat{\delta}\phi(-t) \cdot \mathcal{S}(\delta\phi(t))] = \lim_{t\to 0} \int_{\partial\mathcal{U}} \omega[\delta\phi(t), \hat{\delta}\phi(-t)]. \tag{3.57}$$

Ω may be viewed as a 2-form on $\mathcal{P}_\Sigma$:

$$\Omega[\delta_1\phi, \delta_2\phi] = \lim_{t\to 0} \int_{\partial\mathcal{U}} \omega[\delta_1\phi(t), \delta_2\phi(-t)]. \tag{3.58}$$

[7] We take the convention that each leaf only has one connected component.

[8] There will be some regularity conditions on the field configurations and the operators we insert necessary to make this procedure well-defined. Without these conditions, the field configuration could be arbitrarily singular and distributional. However, in order for the symplectic structure to make sense, its needs to be finite. This, for example, prevents us from allowing configurations which give products of delta functions in the symplectic structure. There will also be other conditions, such as finiteness of energy, which lead to restrictions on the regularity of the field configuration, and we will discuss an example of this later in the chapter. However, at this point the manipulations we are doing are sufficiently general that we do not need to explicitly spell out the regularity conditions. It will suffice to just assume everything is regular enough that things are well-defined, and then impose the right regularity conditions at the end.

This is the symplectic structure.

3.2.4 Comparison to previous approach

Let us hide the $t \to 0$ limit, and write the symplectic structure as[9]

$$\Omega[\delta_1\phi, \delta_2\phi] = \int_{\partial\mathcal{U}} \omega[\delta_1\phi, \delta_2\phi]. \tag{3.59}$$

Recall the claim in Section 1.1 for the value of the symplectic structure:

$$\Omega[\delta_1\phi, \delta_2\phi] = \int_{\Sigma} \omega[\delta_1\phi, \delta_2\phi]. \tag{3.60}$$

The two expressions in (3.59) and (3.60) are clearly very similar. They are both integrals of ω, but over different surfaces. The integral in (3.59) is done over a contour around Σ, while the integral in (3.60) is done over Σ itself.

The reader might wonder whether (3.59) is logically equivalent to (3.60), i.e. whether one of these equations implies the other. This is not the case – the expression in (3.60) is sensitive to the ambiguity (1.46), while the expression in (3.59) is not (since $\partial\mathcal{U}$ has no boundary). Since they do not share this quality, they must be logically distinct. Additionally, the space on which $\delta_1\phi, \delta_2\phi$ exist on is different in each expression. In (3.60), they are tangent vectors to $\mathcal{P}$, the space of on-shell field configurations. On the other hand, in (3.59), they are tangent vectors to $\mathcal{P}_\Sigma$, which consists of all field configurations which can be obtained by applying operators at Σ. Such field configurations only need obey the equations of motion away from Σ.

The fact that (3.59) is insensitive to the ambiguity means that we have done what we set out to do in this section. The symplectic structure, and the theory of the degrees of freedom in the subregion, are now well-defined.

Note that it is possible to obtain an expression from (3.59) that more closely resembles (3.60), and we will now briefly outline how this would work. Suppose $\delta\phi$ is a field variation caused by operator insertions at Σ. By causality, $\delta\phi$ can only have support in $J^+(\Sigma)$, the causal future of Σ. The $\delta_1\phi, \delta_2\phi$ appearing in (3.59) are two such field variations. Hence $\omega[\delta_1\phi, \delta_2\phi]$ also only has support in $J^+(\Sigma)$. Let $\Sigma^+ = J^+(\Sigma) \cap \partial\mathcal{U}$. Naïvely one might write

$$\Omega[\delta_1\phi, \delta_2\phi] = \int_{\partial\mathcal{U}} \omega[\delta_1\phi, \delta_2\phi] = \int_{\Sigma^+} \omega[\delta_1\phi, \delta_2\phi]. \tag{3.61}$$

[9] It is not always safe to assume this expression is valid, because of possible singular behaviour at $t = 0$. Nevertheless, it is at least heuristically useful.

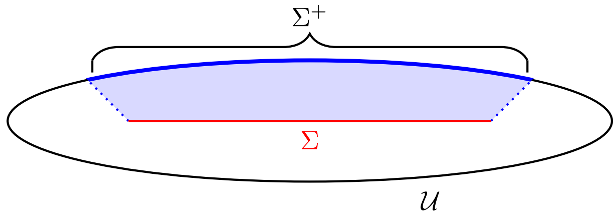


Figure 3.2: The support of $\delta\phi$ on $\partial\mathcal{U}$ is contained in $\Sigma^+ = J^+(\Sigma) \cap \partial\mathcal{U}$.

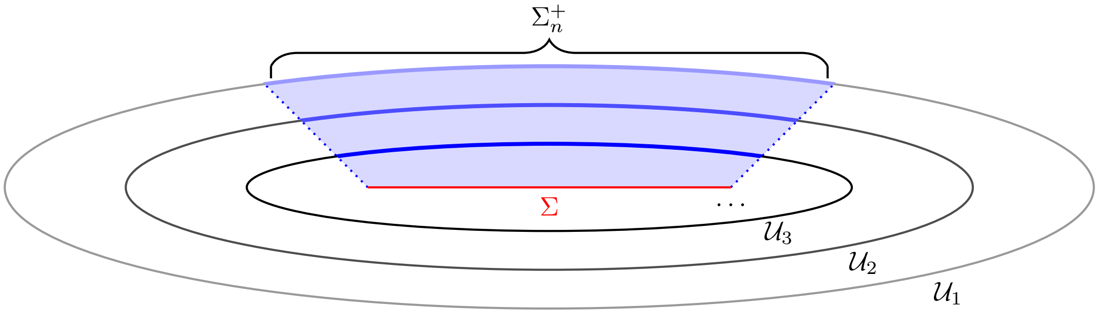


Figure 3.3: A sequence of $\mathcal{U}_n$ such that $\Sigma_n^+ \to \Sigma$.

However, the right-hand side above cannot possibly be correct, because suddenly it is once more subject to the ambiguity (1.46).

The reason this has happened is that we have failed to account for singular and distributional behaviour near $\partial\Sigma^+$, and splitting the integral up in this way only works when the integrand is sufficiently smooth. The proper way to carry out this split must involve some kind of regularisation at $\partial\Sigma$, where one integrates ω against appropriately chosen smooth test functions. The generic result of such a regularisation would be an expression resembling

$$\Omega[\delta_1\phi, \delta_2\phi] = \int_{\Sigma^+} \omega[\delta_1\phi, \delta_2\phi] + \int_{\partial\Sigma^+} X[\delta_1\phi, \delta_2\phi], \tag{3.62}$$

where, under the ambiguity transformation (1.46), X transforms as

$$X[\delta_1\phi, \delta_2\phi] \to X[\delta_1\phi, \delta_2\phi] + \delta_1(\alpha[\delta_2\phi]) - \delta_2(\alpha[\delta_1\phi]) - \alpha[\delta_{12}\phi]. \tag{3.63}$$

Such a transformation rule is necessary to ensure that Ω remains unaffected by the ambiguity. An example of this will be described in Section 3.3.

Once one has obtained (3.62), one can proceed as follows. The expression in (3.59) is valid so long as $\mathcal{U}$ contains Σ. Let us consider a sequence of $\mathcal{U} \in \mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n, \dots$ that contain Σ, and let $\Sigma_n^+ = J^+(\Sigma) \cap \partial\mathcal{U}_n$. The value of the symplectic structure will be independent of n, and one can choose $\mathcal{U}_n$ such that $\Sigma_n^+ \to \Sigma$ as $n \to \infty$. An example is given in Figure 3.3. Once the

$n \to \infty$ limit is taken, (3.62) takes the form

$$\Omega[\delta_1\phi, \delta_2\phi] = \int_\Sigma \omega[\delta_1\phi, \delta_2\phi] + \int_{\partial\Sigma} X[\delta_1\phi, \delta_2\phi], \tag{3.64}$$

where in this expression $\delta_1\phi$ and $\delta_2\phi$ should be evaluated 'just to the future of' Σ.

The expression (3.64) should be viewed as a version of (3.60) for which the correct boundary term has been identified. It would be interesting to evaluate this boundary term directly and see whether it agrees with the boundary terms chosen in other contexts, for example in [71] (although see Section 3.4.1 for a reason that it can't possibly match that boundary term exactly). We leave exploration of this to future work. However, we will comment that it is not obvious that (3.64) has any inherent advantages over (3.59), other than its ease of comparison to previous studies. In fact, it is our view that the contour integral in (3.59) may be the more flexible expression. Whether this is actually true will hopefully become clear in the future.

An alternative equivalent prescription would be to consider a surface $\bar{\Sigma}^+$ that is 'slightly larger' than Σ^+. One could then take the $n \to \infty$ limit, obtaining

$$\Omega = \int_{\bar{\Sigma}} \omega, \tag{3.65}$$

where $\bar{\Sigma}$ is 'slightly larger' than Σ. This would also then be an expression which resolves the ambiguity, which is easy to double check: under $\omega \to \omega + \mathrm{d}K$, we have

$$\Omega = \int_{\bar{\Sigma}^+} \omega \to \int_{\bar{\Sigma}^+} \omega + \underbrace{\int_{\partial\bar{\Sigma}^+} K}_{=0} = \Omega, \tag{3.66}$$

since $\partial\bar{\Sigma}^+$ is outside of the causal future of Σ. This formula for Ω makes it especially clear that the resolution of the ambiguity comes from being able to consistently glue Σ along $\partial\Sigma$ to the rest of spacetime.

Finally, we should note that even though we assumed Σ was a partial Cauchy surface in the above, we can instead just assume that Σ is a bounded set containing no two points which can be connected by a causal curve, i.e. an acausal set. Then the steps in the above derivation all still follow, and (3.59) still applies (so long as $\mathcal{U}$ is chosen to contain Σ). For example, Σ could be a codimension 2 submanifold, or it could not even be a manifold at all. In these cases there is still a notion of the degrees of freedom associated to Σ, and (3.59) provides a symplectic structure for these degrees of freedom. But (3.60) clearly cannot be applied, as there is no well-defined way to integrate ω over such a set. Thus, (3.59) is applicable in a larger range of circumstances than (3.60). However, we should note that it is not clear that the symplectic form is non-trivial in such cases, i.e. we may have $\Omega = 0$. This would indicate that the phase space for the acausal set in question does not exist.

3.2.5 Asymptotic boundaries

So far we have been discussing subregions with finite boundaries, in which case it is always possible to find a $\mathcal{U}$ which encloses the subregion. In the case that Σ has an asymptotic boundary it is not clear that this can be done. However, our results do still apply in this case, subject to the following interpretation.[10]

It is important to recognise that an asymptotic boundary really only makes sense as the limit of a finite boundary as some parameter goes to infinity. For example, in an asymptotically flat spacetime, one can consider a Cauchy surface Σ with an asymptotic boundary at spacelike infinity. But this is only really defined as the limit of Σ_r as $r \to \infty$, where Σ_r is a partial Cauchy surface whose boundary is a sphere of radius r.

Such a prescription for an asymptotic boundary regularises many calculations which would otherwise not be well defined. For example, integrals over Σ should really be considered as integrals over Σ_r, in the limit as $r \to \infty$:

$$\int_\Sigma = \lim_{r\to\infty} \int_{\Sigma_r} . \tag{3.67}$$

Similarly, integrals over the asymptotic boundary should be viewed as integrals over the finite boundary $\partial\Sigma_r$, in the limit as $r \to \infty$:

$$\int_{\partial\Sigma} = \lim_{r\to\infty} \int_{\partial\Sigma_r} . \tag{3.68}$$

Different choices of Σ_r can lead to different integration results. This should not be viewed as an ambiguity in the definition of integration, but rather a dependence of the integral upon the definition of the asymptotic boundary over which it is being performed.

Although it is not always made explicit, this affects (3.60). In particular, the symplectic structure given in (3.60) has an integral over Σ in it, and so contains a limit as $r \to \infty$. To be explicit, we have

$$\int_\Sigma \omega = \lim_{r\to\infty} \int_{\Sigma_r} \omega. \tag{3.69}$$

Armed with this realisation, it is clear how to extend our results to the case of an asymptotic boundary. For each value of r, we can find a $\mathcal{U}_r$ which encloses Σ_r. We can compute the

[10] This interpretation is possibly related to holographic renormalisation [134].

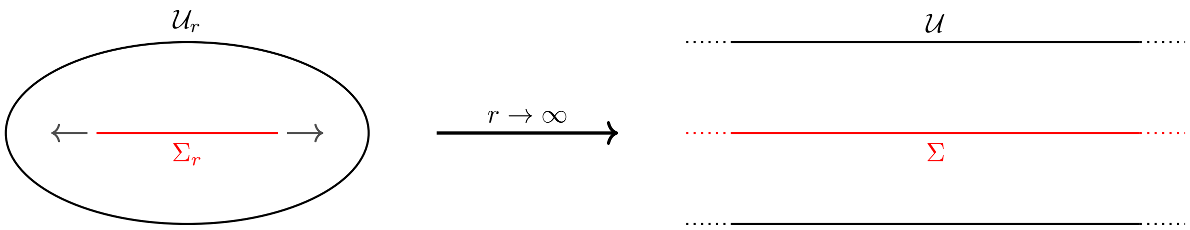


Figure 3.4: For Σ with an asymptotic boundary, the symplectic structure is obtained by integrating over ∂U_r, and then taking the limit as U_r grows to contain the entirety of Σ.

symplectic structure associated with Σ_r as an integral over $\partial\mathcal{U}_r$. Then we can take the limit as $r \to \infty$, obtaining

$$\Omega = \lim_{r\to\infty} \int_{\partial\mathcal{U}_r} \omega. \tag{3.70}$$

Thus we get a well-defined symplectic structure for Σ, even though Σ has an asymptotic boundary.

We should note that to carry out this limit, we technically need to also define a way to 'regularise' the field variations we are considering to finite radius. In other words, we need to be able to take an arbitrary field variation $\delta\phi$ that disobeys the equations of motion on Σ, and convert it to a field variation $\delta\phi_r$ that disobeys the equations of motion only on Σ_r. This should be done so that $\delta\phi \to \delta\phi_r$ as $r \to \infty$, with respect to some topology on the space of fields. Then for each finite value of r, we should evaluate the symplectic structure on these regularised field variations. To be precise, the components of the symplectic structure would be given by

$$\Omega(\delta_1\phi, \delta_2\phi) = \lim_{r\to\infty} \int_{\partial\mathcal{U}_r} \omega(\delta_1\phi_r, \delta_2\phi_r). \tag{3.71}$$

It may be that one cannot find such a finite radius regularisation. Then the formula for the symplectic structure that we are proposing can unfortunately not be used.

3.3 Example: 2D massless scalar

Let us see how our proposed resolution of the ambiguity works in a simple case, namely that of a 2d massless scalar field ϕ in flat space, described by the Lagrangian density

$$L[\phi] = -\frac{1}{2}\,\mathrm{d}\phi \wedge * \mathrm{d}\phi\,. \tag{3.72}$$

Under a variation $\delta\phi$, we have

$$\delta L = -\delta\phi\,\mathrm{d} * \mathrm{d}\phi + \mathrm{d}(\delta\phi * \mathrm{d}\phi)\,, \tag{3.73}$$

so the equations of motion are $\mathrm{d} * \mathrm{d}\phi = 0$, which is just equivalent to $-\Box\phi = (\partial_t^2 - \partial_x^2)\phi = 0$. Also, we get

$$\theta = \delta\phi * \mathrm{d}\phi\,, \quad \omega = \delta_1\phi * \mathrm{d}\delta_2\phi - \delta_2\phi * \mathrm{d}\delta_1\phi\,. \tag{3.74}$$

We choose boundary conditions such that $\phi = 0$ at $t \to -\infty$.

Let us find the phase space for this theory for the subregion Σ defined by $t = 0$, $-1 \leq x \leq 1$. As we have described, this is given by configurations which solve the equations of motion everywhere except at Σ. The most general such configuration is

$$\phi(t, x) = H(t)\big(f(x - t) + g(x + t)\big) + \phi_0(t, x), \tag{3.75}$$

where $H(t)$ is the Heaviside step function, $f(x) = g(x) = 0$ for $|x| > 1$, and $\phi_0(t, x) = 0$ for $t \neq 0, |x| > 1$. In general, f, g, ϕ_0 can be distributions instead of just plain functions.

The old (ambiguous) symplectic structure would be

$$\Omega_{\text{old}} = \int_\Sigma \omega. \tag{3.76}$$

Of course, this expression suffers from the ambiguity at $\partial\Sigma$. Actually, ϕ can be singular at Σ, so we should actually define the old structure in terms of a surface Σ_ϵ just to the future of Σ, at $t = \epsilon > 0$, $-1 \leq x \leq 1$. Then the old symplectic structure can be written

$$\Omega_{\text{old}} = \lim_{\epsilon \to 0} \int_{\Sigma_\epsilon} \omega. \tag{3.77}$$

This doesn't fix the ambiguity.

To fix the ambiguity, we can use our new formula

$$\Omega = \int_{\partial\mathcal{U}} \omega. \tag{3.78}$$

Let's pick $\mathcal{U}$ such that $\partial\mathcal{U}$ consists of four parts:

$$\Sigma_\epsilon = \{t = \epsilon, -1 \leq x \leq 1\}, \tag{3.79}$$

$$\mathcal{B}_1 = \{t - x = 1 + \epsilon, -1 - \epsilon \leq t + x \leq -1 + \epsilon\}, \tag{3.80}$$

$$\mathcal{B}_2 = \{t + x = 1 + \epsilon, -1 - \epsilon \leq t - x \leq -1 + \epsilon\}, \tag{3.81}$$

and $\mathcal{C}$, which closes $\partial\mathcal{U}$ in the past of Σ. This is depicted in Figure 3.5. Since $\phi|_{\mathcal{C}} = 0$, we have

$$\Omega = \int_{\Sigma_\epsilon} \omega + \int_{\mathcal{B}_1} \omega + \int_{\mathcal{B}_2} \omega. \tag{3.82}$$

Now since this expression remains valid for any choice of Σ, we can take the limit $\epsilon \to 0$. Thus,

$$\Omega = \Omega_{\text{old}} + \lim_{\epsilon \to 0} \left(\int_{\mathcal{B}_1} \omega + \int_{\mathcal{B}_2} \omega \right). \tag{3.83}$$

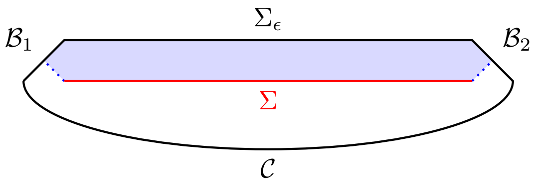


Figure 3.5: The choice of $\partial\mathcal{U}$ we are using to evaluate the symplectic structure for a 2d scalar field in the subregion $t = 0$,$-1 \leq x \leq 1$.

In the $\epsilon \to 0$ limit, the latter terms only depend on the fields arbitrarily close to the boundary $\partial\Sigma$. Therefore these terms represent the correct boundary corrections to Ω required to fix the ambiguity.

It is useful to explicitly evaluate the boundary terms using (3.74) and the explicit field configurations (3.75). We have

$$\int_{\mathcal{B}_1} \theta = \int_{\mathcal{B}_1} \delta\phi * \mathrm{d}\phi \tag{3.84}$$

$$= \int_{-1-\epsilon}^{-1+\epsilon} \mathrm{d}x^+ \, \delta\phi \partial_{x^+}\phi \Big|_{x^- = -1-\epsilon} \tag{3.85}$$

$$= \int_{-1-\epsilon}^{-1+\epsilon} \mathrm{d}x^+ \, (\delta f(-1-\epsilon) + \delta g(x^+))\partial_{x^+} g(x^+) \tag{3.86}$$

$$= \int_{-1-\epsilon}^{1+\epsilon} \mathrm{d}x^+ \, \delta g(x^+)\partial_{x^+} g(x^+), \tag{3.87}$$

where $x^\pm = x \pm t$. Similarly,

$$\int_{\mathcal{B}_2} \theta = \int_{\mathcal{B}_2} \delta\phi * \mathrm{d}\phi \tag{3.88}$$

$$= \int_{1-\epsilon}^{1+\epsilon} \mathrm{d}x^- \, \delta\phi \partial_{x^-}\phi \Big|_{x^+ = 1+\epsilon} \tag{3.89}$$

$$= \int_{1-\epsilon}^{1+\epsilon} \mathrm{d}x^- \, (\delta f(x^-) + \delta g(1+\epsilon))\partial_{x^-} f(x^-) \tag{3.90}$$

$$= \int_{1-\epsilon}^{1+\epsilon} \mathrm{d}x^+ \, \delta f(x^-)\partial_{x^-} f(x^-). \tag{3.91}$$

Therefore

$$\Omega = \Omega_{\text{old}} + \lim_{\epsilon \to 0} \left(\int_{-1-\epsilon}^{-1+\epsilon} \mathrm{d}x \, \delta_2 g(x) \partial_x \delta_1 g(x) + \int_{1-\epsilon}^{1+\epsilon} \delta_2 f(x) \partial_x \delta_1 f(x) - (1 \leftrightarrow 2) \right), \tag{3.92}$$

where

$$\Omega_{\text{old}} = \lim_{\epsilon \to 0} \int_{\Sigma_\epsilon} \left(\delta_1\phi * \mathrm{d}\delta_2\phi - \delta_2\phi * \mathrm{d}\delta_1\phi \right). \tag{3.93}$$

The extra terms can be non-vanishing because of possible singular behaviour in $\delta f(x)$ at $x = 1$ and $\delta g(x)$ at $x = -1$.

Suppose we have $\delta_1 g(x) = \delta(x+1)$ and $\delta_2 g(x) = H(1+x)H(1-x)$, while $\delta_{1,2} f = 0$. Then

$$\Omega = \lim_{\epsilon\to 0}\int_{-1-\epsilon}^{-1+\epsilon}\Big(H(1+x)\partial_x\delta(1+x) - \delta(1+x)\partial_x H(1+x)\Big) = -2\delta(0) = \infty? \tag{3.94}$$

There is no contribution at Σ_ϵ, because $\delta_1\phi$ is zero there ($g(x)$ is left moving). So the symplectic form would appear to diverge for these field variations, which is a clearly a problem! At best, this would imply such field variations are non-physical, but at worst it makes the whole construction mathematically inconsistent. The way to fix this, and other similar divergences, is to recall that the true definition of the symplectic form (3.58) involves time displacement. If we included this from the start, we would have found

$$\Omega = \Omega_{\text{old}} + \lim_{\epsilon\to 0}\lim_{t\to 0}\bigg(\int_{-1-\epsilon}^{-1+\epsilon}\mathrm{d}x\,\delta_2 g(x+t)\partial_x\delta_1 g(x-t) + \int_{1-\epsilon}^{1+\epsilon}\delta_2 f(x-t)\partial_x\delta_1 f(x+t) - (1\leftrightarrow 2)\bigg). \tag{3.95}$$

Then setting $\delta_1 g(x) = \delta(x+1)$ and $\delta_2 g(x) = H(1+x)H(1-x)$ gives

$$\Omega = \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{-1-\epsilon}^{-1+\epsilon}\Big(H(1+x+t)\partial_x\delta(1+x-t) - \delta(1+x-t)\partial_x H(1+x+t)\Big) \tag{3.96}$$

$$= -2\lim_{\epsilon\to 0}\lim_{t\to 0}\int_{-1-\epsilon}^{-1+\epsilon}\delta(1+x+t)\delta(1+x-t) = 0. \tag{3.97}$$

Thus, Ω is once more well-defined.

On the other hand, this time-displacement regularisation does not set the extra terms to zero for all possible field variations. Suppose $\delta_1 g(x) = \delta(x+1)$, $\delta_2 g(x) = A(x)H(1+x)H(1-x)$ and $\delta_{1,2} f = 0$, where $A(x)$ is some arbitrary function. Then we have

$$\Omega = \lim_{\epsilon,t\to 0}\int_{-1-\epsilon}^{-1+\epsilon}\Big(A(x+t)H(1+x+t)\partial_x\delta(1+x-t) - \delta(1+x-t)\partial_x(A(x+t)H(1+x+t))\Big) \tag{3.98}$$

$$= -2A'(-1). \tag{3.99}$$

Let us discuss a final point before moving on from this example. In physical situations, it is often natural to restrict to field configurations with finite energy, and this will impose certain continuity conditions on the fields. One might optimistically hope that such a restriction assists in resolving the boundary ambiguity in the old formula for the symplectic structure. However this is not true. We can demonstrate this explicitly in the example we are studying, for which the energy density is

$$\frac{1}{2}\Big((\partial_x\phi)^2 + (\partial_t\phi)^2\Big). \tag{3.100}$$

For finite energy, it would be sufficient to have $\partial_x\phi$ and $\partial_t\phi$ be square-integrable. Suppose we changed $\omega\to\omega + \mathrm{d}K$, so that

$$\Omega_{\text{old}} \to \Omega_{\text{old}} + \lim_{\epsilon\to 0}\int_{\partial\Sigma_\epsilon} K. \tag{3.101}$$

Ω itself would remain unchanged, because the integrals over $\mathcal{B}_1, \mathcal{B}_2$ in (3.83) would cancel the latter term above. Moreover, despite the requirement of square integrability, the derivatives of ϕ can be discontinuous. So if (for example)

$$K = \partial_x \delta_1 \phi \partial_x^2 \delta_2 \phi - \partial_x \delta_2 \phi \partial_x^2 \delta_1 \phi, \tag{3.102}$$

then Ω_{old} would genuinely change its value, and so be ambiguous, even when restricting to finite energy configurations.

For more general field theories, a similar thing will happen. Namely, the finiteness of energy will impose some conditions on the degree of continuity of the fields $\partial\Sigma$. But these conditions will not be enough by themselves to fix the ambiguity, which will still be present in the old formula $\int_\Sigma \omega$. The ambiguity is only fixed by using the new formula $\int_{\partial\mathcal{U}} \Sigma$.

3.4 Discussion

In this chapter we have shown that the symplectic structure associated to a subregion with partial Cauchy surface Σ should be written as the contour integral of the form ω around Σ. This is in contrast to the previous notion that the symplectic structure should be written as the integral of ω over Σ itself, and, as we have discussed, resolves the boundary ambiguities inherent to that belief. There are a number of other immediate consequences of our results, a couple of which we will briefly describe below.

3.4.1 Gauge symmetries

Suppose that the field theory we are interested in has a gauge symmetry, and consider a gauge transformation $\phi \to \phi + \delta_\lambda \phi$, where λ is some parameter. As we explained in Section 1.1, when the equations of motion are obeyed, $\omega[\delta\phi, \delta_\lambda\phi]$ is an exact form, so let us write $\omega[\delta\phi, \delta_\lambda\phi] = \mathrm{d}(\oint q_\lambda[\delta\phi])$. If we assume that the symplectic structure is given by $\Omega = \int_\Sigma \omega$, and that the phase space consists of solutions to the equations of motion, then we clearly have

$$\Omega[\delta\phi, \delta_\lambda\phi] = \int_\Sigma \omega[\delta\phi, \delta_\lambda\phi] = \int_{\partial\Sigma} \oint q_\lambda, \tag{3.103}$$

and in general, the integral on the right-hand side will not vanish. The implication is that there are some gauge transformations which do not correspond to degenerate directions of the symplectic structure. Such gauge transformations have non-trivial action at $\partial\Sigma$, and are referred to as large.

However, as we have shown, the correct expression for the symplectic structure is $\Omega = \int_{\partial\mathcal{U}} \omega$. The correct phase space only includes configurations which disobey the equations of motion at Σ, and $\Sigma \cap \partial\mathcal{U} = \emptyset$, so the equations of motion are obeyed at $\partial\mathcal{U}$. Therefore, we have

$$\Omega[\delta\phi, \delta_\lambda\phi] = \int_{\partial\mathcal{U}} \mathrm{d}(\not{\delta} q_\lambda) = 0, \tag{3.104}$$

where the last equality holds because $\partial\mathcal{U}$ has no boundary. Therefore, we can unequivocally state that *all* gauge transformations are non-physical, even ones which are non-trivial at $\partial\Sigma$. This much more closely fits our intuition for what a gauge transformation is.

We want to emphasise that we do not claim that this result invalidates previous work on large gauge transformations. Rather, we take the view that it should change the interpretation of that work. To be clear what we mean, suppose that there is a H_λ such that

$$\delta H_\lambda = \int_{\partial\Sigma} \not{\delta} q_\lambda. \tag{3.105}$$

If one assumes that $\Omega = \int_\Sigma \omega$, then the gauge transformation corresponding to λ is integrable, and generated by H_λ. Suppose however that instead we use the correct symplectic structure $\Omega = \int_{\partial\mathcal{U}} \omega$. Then H_λ can still be thought of as the generator of some field transformation $\phi \to \phi + \tilde{\delta}_\lambda\phi$ obeying

$$\Omega[\delta\phi, \tilde{\delta}_\lambda\phi] = \int_{\partial\mathcal{U}} \omega[\delta\phi, \tilde{\delta}_\lambda\phi] = \delta H_\lambda. \tag{3.106}$$

Clearly, $\tilde{\delta}_\lambda\phi$ cannot be merely a gauge transformation, since we have just shown that all gauge transformations are degenerate in the symplectic structure. We will refer to $\tilde{\delta}_\lambda\phi$ as a pseudo-gauge transformation, due to its subtle similarity with a true gauge transformation[11]. We believe that the pseudo-gauge transformations are worth studying, and suspect that many of the results regarding large gauge transformations should instead be interpreted as applying to pseudo-gauge transformations.

Let us make some general comments. When one first encounters large gauge transformations, it is understandable to be surprised at their claimed physical significance. After all, the standard story about gauge transformations is that they reflect a redundancy in the field description, and so should not be thought of as physical. However, looking further, one begins to appreciate that if the gauge transformation is non-trivial at the boundary, all kinds of strange things can happen. For example, boundary conditions can be modified or even violated by a large gauge

[11] It would be interesting to compare these pseudo-gauge transformations to the 'would-be' gauge transformations of [32].

transformation, and total derivatives which are usually neglected in the action can suddenly start to play a role. With these observations in mind, it becomes easier to accept that large gauge transformations which are non-trivial at an asymptotic boundary, or a true boundary of spacetime, can physically change the state.

However, it is much less obvious that a large gauge transformation *in a subregion* should have any physical meaning. After all, the boundary of a subregion is really nothing special. There are no boundary conditions there, and total derivatives in the action only give boundary terms at the boundary of the entire spacetime, not at the boundary of any arbitrary subregion inside of it. Thus, it is perhaps a relief that the symplectic form we find here indicates that large gauge transformations in a subregion actually are non-physical.

Does Section 3.2.5 now imply that even those large gauge transformations which act on a whole spacetime are also non-physical? This is less certain. In order to be able to apply the results of this section with the results of this section, we need a consistent regularisation of large gauge transformations at finite r (in the sense of (3.71)), and it is not clear that such a regularisation exists. Indeed, we have indications from other contexts that these large gauge transformations are physical. Thus, it seems plausible that such a regularisation does *not* exist.

One thing more that we should mention is that the results of this chapter only apply to non-gravitational theories. This is because everything depends on the existence of local observables at Σ. In a gravitational theory (i.e. one with diffeomorphism invariance), no such local observables exist. This means that everything we have described in this section can only apply to gauge transformations which are not diffeomorphisms. As far as this chapter is concerned, the jury is therefore still out on whether large diffeomorphisms in subregions are physical.

3.4.2 Dependence on fields outside the subregion?

The resolution of the ambiguity that we propose $\Omega = \int_{\partial\mathcal{U}} \omega$ appears to depend on the values taken by fields away from the subregion. One might wonder whether this means that we have not actually resolved the ambiguity at all. Could it be that we have actually just rewritten the ambiguity in terms of a dependence on the state outside of Σ?

The answer is no – we have genuinely fixed the ambiguity, subject to a certain caveat. To see this, note again that the choice of $\mathcal{U}$ is arbitrary, so long as it contains Σ, and the equations of motion are obeyed away from Σ. Thus, we can make $\mathcal{U}$ arbitrarily small, and the symplectic form we propose can only depend on fields arbitrarily close to Σ. One might say that this doesn't

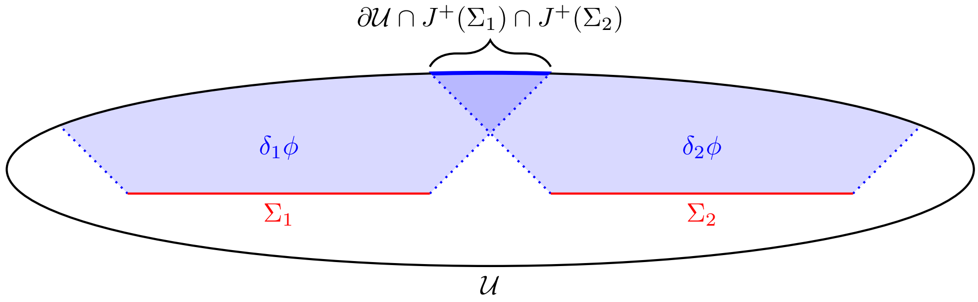


Figure 3.6: The joint support of $\delta_1\phi, \delta_2\phi$ on $\partial\mathcal{U}$ is contained in $\partial\mathcal{U} \cap J^+(\Sigma_1) \cap J^+(\Sigma_2)$.

actually get rid of the issue just noted – after all a dependence on fields arbitrarily close to Σ still includes a dependence on fields *outside* of Σ.

However, we would argue that this should not be seen as a problem. Instead it should be seen as a hint at what the 'degrees of freedom in a subregion' actually means. It is telling us that if a quantity depends only on the fields in an arbitrarily small open neighbourhood of Σ, then it *is* a degree of freedom on Σ. An obvious example of why this is correct is the case of a scalar field ϕ. If ∂_t is normal to Σ, then $\partial_t\phi$ is clearly a degree of freedom on Σ, but to evaluate $\partial_t\phi$ we need to know the value of ϕ in an open neighbourhood of Σ. The same goes for any derivative of ϕ at $\partial\Sigma$ in a direction which is not tangent to $\partial\Sigma$.

So the caveat is that we need to define degrees of freedom 'on' Σ in this way. If we do not, we cannot resolve the ambiguity at all.

3.4.3 Correlations between distinct subregions

Consider the subregions associated to two distinct and spatially separated partial Cauchy surfaces Σ_1, Σ_2. Let us ask the following question: can an observation in one of these subregions affect an observation in the other? Let A_1, A_2 be observables on Σ_1, Σ_2, generating field variations $\delta_1\phi, \delta_2\phi$ respectively. We can answer our question by calculating the Poisson bracket of A_1 and A_2, which is equal to $\Omega[\delta_1\phi, \delta_2\phi]$.

There are two cases to consider. In the first case, we assume there is a 'gap' between Σ_1 and Σ_2. Let us pick a $\mathcal{U}$ as in Figure 3.6. By causality, the supports of $\delta_1\phi, \delta_2\phi$ must be contained in $J^+(\Sigma_1), J^+(\Sigma_2)$ respectively. Therefore, the integrand of $\Omega[\delta_1\phi, \delta_2\phi] = \int_{\partial\mathcal{U}} \omega[\delta_1\phi, \delta_2\phi]$ can only be supported in $\partial\mathcal{U} \cap J^+(\Sigma_1) \cap J^+(\Sigma_2)$.

We could at this point try to compute this integral directly, but it turns out to be much easier to instead just pick a different $\mathcal{U}$ such that $\partial\mathcal{U} \cap J^+(\Sigma_1) \cap J^+(\Sigma_2) = \emptyset$, as in Figure 3.7.

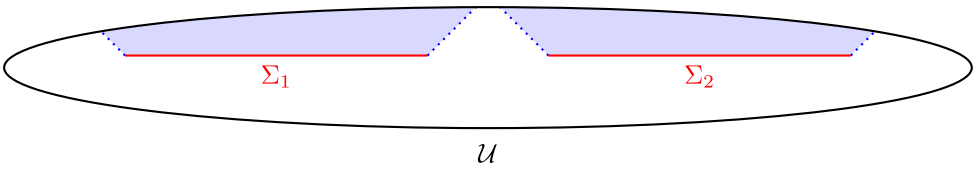


Figure 3.7: With this choice of $\mathcal{U}$, we have $\partial\mathcal{U} \cap J^+(\Sigma_1) \cap J^+(\Sigma_2) = \emptyset$.

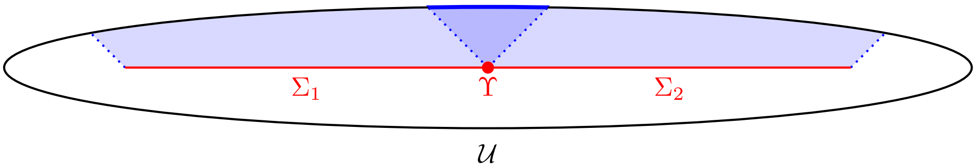


Figure 3.8: When there is no gap, it is not possible to choose $\mathcal{U}$ such that $\partial\mathcal{U} \cap J^+(\Sigma_1) \cap J^+(\Sigma_2) = \emptyset$. The red dot denotes the edge Υ joining Σ_1 and Σ_2, commonly known as the entangling surface.

This choice is made possible by the gap between Σ_1 and Σ_2. The support of the integrand in $\Omega[\delta_1\phi, \delta_2\phi] = \int_{\partial\mathcal{U}} \omega[\delta_1\phi, \delta_2\phi]$ is empty, so $\Omega[\delta_1\phi, \delta_2\phi] = 0$. Therefore, the Poisson bracket of A_1 and A_2 vanishes, and we may conclude that no observation on Σ_1 can affect an observation on Σ_2.

The situation changes when there is no gap between Σ_1 and Σ_2, which is the second case we consider, and is shown in Figure 3.8. It is no longer possible in such circumstances to choose $\mathcal{U}$ such that $\partial\mathcal{U} \cap J^+(\Sigma_1) \cap J^+(\Sigma_2) = \emptyset$, so we can not use the above trick to show that $\Omega[\delta_1\phi, \delta_2\phi] = 0$. In fact, it sometimes is possible to find A_1 and A_2 such that $\{A_1, A_2\} = \Omega[\delta_1\phi, \delta_2\phi] \neq 0$. Hence there *are* observations on Σ_1 which can affect observations on Σ_2, even though these two surfaces are distinct and spatially separated.

To give an example, let us return to the set up in Section 3.3 of a 2D massless scalar field. About a background $\phi = 0$, consider the field variations

$$\delta_1\phi = H(t)H(t-x)H(1+x-t), \tag{3.107}$$

$$\delta_2\phi = H(t)H(t+x)H(1-x-t). \tag{3.108}$$

These obey the equations of motion everywhere except at $\Sigma_1 = \{t = 0, -1 \leq x \leq 0\}$, $\Sigma_2 = \{t = 0, 0 \leq x \leq -1\}$ respectively, and so may be thought of being sourced by operators A_1, A_2 inserted on these surfaces. If we displace Σ_1 to $t = -\epsilon$ and Σ_2 to $t = \epsilon$ for small $\epsilon > 0$, the field variations become

$$\delta_1\phi(-\epsilon) = H(t-\epsilon)H(t-\epsilon-x)H(1+x-t+\epsilon), \tag{3.109}$$

$$\delta_2\phi(\epsilon) = H(t+\epsilon)H(t+\epsilon+x)H(1-x-t-\epsilon). \tag{3.110}$$

Then, using (3.52), we can write the Poisson bracket of A_1 and A_2 as

$$\{A_1, A_2\} = -\lim_{\epsilon\to 0}\int_{\partial\mathcal{U}} \omega[\delta_1\phi(-\epsilon), \delta_2\phi(\epsilon)]. \tag{3.111}$$

Here $\mathcal{U}$ is any surface which contains Σ_1 and Σ_2 (including any time-displacement). For convenience, we can choose $\mathcal{U}$ such that $\partial\mathcal{U}$ contains $\{t = 1, -3 \leq x \leq 3\}$, and such that $\delta_1\phi(-\epsilon) = \delta_2\phi(\epsilon) = 0$ on the rest of $\partial\mathcal{U}$ (for sufficiently small ϵ). Then we have

$$\{A_1, A_2\} = -\lim_{\epsilon\to 0}\int_{-3}^{3} \mathrm{d}x\,\left(\delta_1\phi(-\epsilon)\partial_t\delta_2\phi(\epsilon) - \delta_2\phi(\epsilon)\partial_t\delta_1\phi(-\epsilon)\right)\Big|_{t=1}. \tag{3.112}$$

Now, for $t > 0$ (and $\epsilon < 1$) we may write

$$\partial_t\delta_1\phi(-\epsilon) = \delta(t-\epsilon-x) - \delta(1+x-t+\epsilon), \tag{3.113}$$

$$\partial_t\delta_2\phi(\epsilon) = \delta(t+\epsilon+x) - \delta(1-x-t-\epsilon), \tag{3.114}$$

and using these in (3.112) yields

$$\begin{aligned}\{A_1, A_2\} = -\lim_{\epsilon\to 0}\Big[&H(1-\epsilon)\big(H(-2\epsilon)H(-1) - H(1)H(2\epsilon)\big)\\ &- H(1+\epsilon)\big(H(2\epsilon)H(-1) - H(1)H(-2\epsilon)\big)\Big].\end{aligned} \tag{3.115}$$

Since $\epsilon > 1$, the final result is

$$\{A_1, A_2\} = 1. \tag{3.116}$$

So this gives an example of two observables in spatially distinct regions whose Poisson bracket does not vanish.

At first sight, the fact that $\{A_1, A_2\} \neq 0$ appears to be a significant problem, because it implies that faster than light communication can take place. This would clearly be an issue in any relativistic theory with a self-consistent causal structure. However, after some more thought, it is quite plausible that this problem is not as big as one might initially think. In order for $\{A_1, A_2\} \neq 0$, we need A_1 and A_2 to both have support arbitrarily close to the common part of the boundary $\partial\Sigma_1 \cap \partial\Sigma_2$ – if they did not then we could simply redefine Σ_1, Σ_2 to be slightly shrunken so there is a gap between them, and the previous argument that $\{A_1, A_2\} = 0$ would follow. So really, the distance between A_1 and A_2 would be vanishingly small, which means that the spacelike trajectories of any 'faster than light' communication would be far too short to be in any way problematic.

Another way to look at this uses what we pointed out in the previous subsection: for our disambiguation to be consistent, the degrees of freedom associated with a partial Cauchy surface Σ need to depend on the fields in an arbitrarily small open neighbourhood of Σ. For Σ_1, Σ_2 without a gap, any open neighbourhood of Σ_1 will necessarily have non-trivial overlap with any open neighbourhood of Σ_2. Thus, there are degrees of freedom (depending on the fields in the overlap) which should be thought of as being associated with *both* Σ_1 and Σ_2. When we compute a non-zero $\{A_1, A_2\}$, we are probing these overlap degrees of freedom.

These shared degrees of freedom depend on the fields in an arbitrarily small open neighbourhood of $\partial\Sigma_1 \cap \partial\Sigma_2$, and so should be thought of as living on this codimension 2 surface. Such degrees of freedom are called 'edge modes'. Their existence implies interesting things about the structure of the phase space of the classical theory, and of the Hilbert space of the classical theory. If there were no edge modes, then we could factorise these spaces as

$$\mathcal{P}_{12} = \mathcal{P}_1 \times \mathcal{P}_2, \qquad \mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{3.117}$$

Here, $\mathcal{P}_{12}, \mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{H}_{12}, \mathcal{H}_1, \mathcal{H}_2$ are the phase spaces and Hilbert spaces of $\Sigma_1 \cup \Sigma_2, \Sigma_1, \Sigma_2$ respectively. However, when there are edge modes, this simple factorisation would result in an overcounting of degrees of freedom – the shared degrees of freedom would be accounted for in both factors. To get the true phase space and Hilbert space, we have to take some subspace of the product, i.e.

$$\mathcal{P}_{12} \subset \mathcal{P}_1 \times \mathcal{P}_2, \qquad \mathcal{H}_{12} \subset \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{3.118}$$

In each case, the subspace is defined by the requirement that the edge modes match for the two surfaces Σ_1, Σ_2.

Another, potentially more precise, way to notate this would be to first fix the values of the edge modes, and then construct $\mathcal{P}_1, \mathcal{P}_2, \mathcal{H}_1, \mathcal{H}_2$. In each case, $\mathcal{P}_{12}, \mathcal{H}_{12}$ would now be exactly given by products, because all of the edge modes are fixed. Then to recover the true $\mathcal{P}_{12}, \mathcal{H}_{12}$, we would have to sum over the possible values of the edge modes, i.e.

$$\mathcal{P}_{12} = \bigcup_\alpha \mathcal{P}_1^\alpha \times \mathcal{P}_2^\alpha, \qquad \mathcal{H}_{12} = \bigoplus_\alpha \mathcal{H}_1^\alpha \otimes \mathcal{H}_2^\alpha. \tag{3.119}$$

Here, α is an index which is supposed to span over all possible configurations of the edge modes, and $\mathcal{P}_1^\alpha, \mathcal{P}_2^\alpha, \mathcal{H}_1^\alpha, \mathcal{H}_2^\alpha$ are phase spaces and Hilbert spaces for each fixed edge mode configuration α.

In the classical case, it might be beneficial to think of fixing the edge modes as setting some boundary conditions. The sum over edge modes would then be a sum over boundary conditions. There are potentially different ways we could carry out this sum – our prescription for the total symplectic structure tells us which is the right one. Essentially, we want the result of the sum to reproduce the symplectic structure we have found. In [75], the ambiguity in the covariant phase space was resolved in the presence of boundary conditions. It would be interesting to see whether the results of that paper match what we have found here, when the edge modes are fixed.

In the quantum case, this structure in the Hilbert space is reflected in entanglement between the quantum states in the two subregions. Thus, we are led to believe that the ideas described here could enable a quantitative description of the entanglement configuration between the two subregions in terms of emergent degrees of freedom living at $\partial\Sigma_1 \cap \partial\Sigma_2$ (which is sometimes known as the entangling surface). This is a topic that has been the subject of much recent interest – see [41, 42, 58, 62, 73, 74, 95, 105, 112, 128, 129, 136, 149, 150], amongst many others. The details of this description obviously need to be worked out, but we are hopeful that our results provide some useful mathematical tools for analysing entanglement in the classical limit.

Chapter 4

The Holographic Dual of the Entanglement Wedge Symplectic Form

4.1 Introduction

Most research stemming from the discovery of the AdS/CFT correspondence [115, 158] can be loosely sorted into two categories. The first involves using the duality to translate a hard question about quantum field theory into an easier one about gravity, or vice-versa. This translation makes use of the so-called holographic dictionary, i.e. the collection of 1-to-1 maps between concepts in the bulk gravity theory and the boundary field theory. But many pages of the dictionary remain empty, and the second category of research endeavours to fill these pages with new entries, in order to both deepen our understanding of holography, and widen the scope for its potential applications. In recent years a coherent picture of a particular section of the dictionary, under the heading 'subregion duality', has emerged [8, 35, 36, 52, 59, 64, 67, 87, 95, 98, 129, 148]. The entries in this section make precise the relationship between boundary locality and bulk locality by identifying properties of a given subregion of the boundary with those of an associated subregion of the bulk. The current consensus is that the bulk dual of a boundary subregion with Cauchy surface A is its 'entanglement wedge', which is the domain of dependence of a codimension 1 surface in the bulk joining A with its Hubeny-Rangamani-Ryu-Takayanagi (HRT) surface (i.e. the codimension 2 surface homologous to A in the bulk with extremal area). The standard depiction of the entanglement wedge is given in Figure 4.1.

This chapter makes an argument for a new entry in this section of the dictionary. To explain our new entry, consider the classical large N limit of the bulk gravity theory. Such a limit should

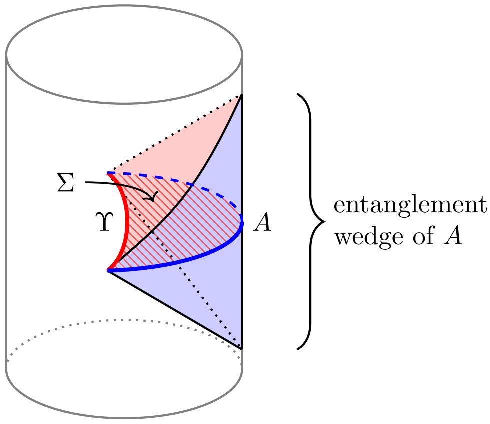


Figure 4.1: The entanglement wedge of a boundary subregion A is defined as the domain of dependence of a partial Cauchy surface Σ interpolating between A and its associated HRT surface Υ. (The colour scheme here will be used throughout this chapter. Blue colouring indicates something on the boundary, whereas red colouring indicates something in the bulk.)

permit a classical Hamiltonian description, including a phase space whose points correspond to the different possible classical field configurations. A popular and versatile construction of the classical phase space of a theory of fields is the covariant phase space formalism described in Section 1.1. Unfortunately that formalism has the well-known ambiguity in the presence of boundaries that we also described in that section.

In the case where Σ is a complete Cauchy surface for a bulk asymptotically AdS spacetime in a holographic theory, the boundary dual to Ω has recently been understood in terms of the Berry curvature [5, 31, 123, 131, 133, 157] of the boundary Hilbert space. To remind the reader of the definition of Berry curvature, consider a closed curve $C : S^1 \to \mathcal{H}$ of normalised states in a Hilbert space $\mathcal{H}$, and suppose we choose a sequence of n states ordered along this curve, $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$. Consider a limit in which $n \to \infty$ and the states $|\psi_i\rangle$ densely cover the curve C, as shown in Figure 4.2. Then one may show that

$$\langle\psi_1|\psi_n\rangle \langle\psi_n|\psi_{n-1}\rangle \ldots \langle\psi_3|\psi_2\rangle \langle\psi_2|\psi_1\rangle \longrightarrow \exp(i\gamma), \tag{4.1}$$

where $\gamma = \oint_C a$, and

$$a = i \langle\psi|\mathrm{d}|\psi\rangle \tag{4.2}$$

is a real 1-form on Hilbert space. In other words, upon traversing the curve C, the state of the system picks up a phase shift given by γ. This is the Berry phase.[1] The map $|\psi\rangle \to e^{if} |\psi\rangle$,

[1] The original definition of Berry phase in terms of the eigenstates of a slowly varying Hamiltonian is a special case

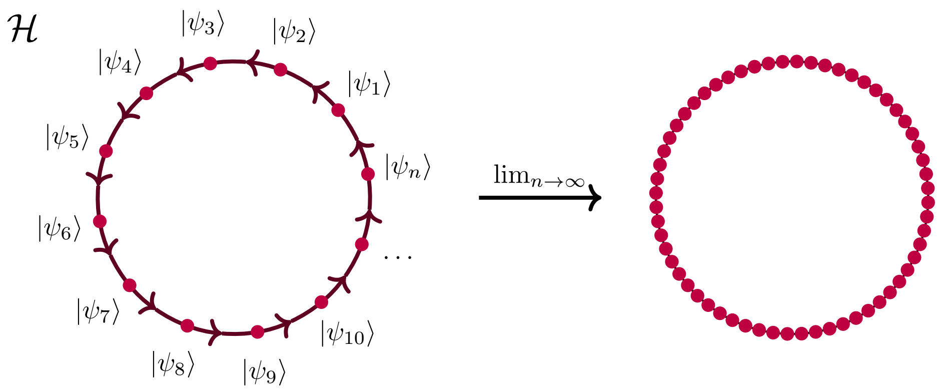


Figure 4.2: To find the Berry phase of a curve of normalised states in a Hilbert space $\mathcal{H}$, one picks a sequence of states $|\psi_i\rangle$ along that curve, and computes the product of the successive transition amplitudes between these states. One then takes the limit in which the sequence of states densely covers the curve.

where f is a real function on Hilbert space, is a gauge transformation that leaves the Berry phase unchanged. However, under this transformation we do have $a \to a - \mathrm{d}f$. In other words a transforms like a $U(1)$ connection; it is called the Berry connection. The curvature of the Berry connection (called the Berry curvature) is gauge-invariant, and is given by the formula

$$\mathrm{d}a = i\,\mathrm{d}\langle\psi| \wedge \mathrm{d}|\psi\rangle\,. \tag{4.3}$$

Returning to the holographic context, one may construct boundary states $|\lambda\rangle$ by inserting operators in a Euclidean path integral. The parameters λ are the coefficients of these operator insertions, and set the boundary conditions for the bulk fields; in the classical limit there is a 1-to-1 map between the boundary conditions λ and bulk field configurations ϕ. It was shown in [27, 28] that the bulk symplectic form is equal to the pullback of the boundary Berry curvature through this map. In light of subregion duality, an immediate question presents itself: is there a generalisation of this result to subregions? The purpose of this chapter is to answer this question in the affirmative.

On the boundary side, we consider states $|\lambda\rangle$ reduced to a fixed subregion A. To be precise, this means the reduced density matrix

$$\rho(\lambda) = \mathrm{tr}_{\bar{A}}\,|\lambda\rangle\langle\lambda|\,, \tag{4.4}$$

where $\mathrm{tr}_{\bar{A}}$ denotes a trace over the part of the boundary Hilbert space containing the degrees of freedom in $\bar{A}$, the complement of A. Because of entanglement between A and $\bar{A}$, $\rho(\lambda)$ is in

of the one given here. This simpler and more general definition is sufficient for our purposes.

general a mixed state, but Berry phases are only defined for pure states. Thus, we will need a generalisation of Berry phase. The generalisation we use is due to Uhlmann [38, 145–147], and is based on a maximisation of transition probabilities between purifications of density matrices. This leads to a notion of holonomy in the space of purifications, and will be described in more detail in Section 4.2. For now, suffice it to say that to any closed curve of density matrices of the above form one may associate a phase shift, which we will refer to as the Uhlmann phase, and which may be written in terms of the integral of a connection around that curve. The curvature of this connection is the boundary quantity that we are interested in.

On the bulk side, one may compute the symplectic form Ω of the entanglement wedge of A, using the covariant phase space recipe. Such a symplectic form is subject to the boundary ambiguity mentioned above at the HRT surface. There is a particular way to resolve this ambiguity which will be described in this chapter.

Our claim is that this now unambiguous entanglement wedge symplectic form is exactly dual to the curvature of the Uhlmann phase of A. This generalises in a very natural way the result of [27, 28]. At the same time, it fulfils a more general principle for resolving the boundary ambiguity in the symplectic form. In a certain sense, the boundary ambiguity is representative of the following fact. When one divides space into two subregions joined by a common boundary, one must make a choice about the degrees of freedom that lie on that boundary. In particular, one must decide which of the two subregions each such degree of freedom should be associated with, and in principle, without additional constraints, one is free to make this decision however one likes.[2] However, the holographic context is an additional constraint. There is only one way to sort the degrees of freedom on the boundary in a way that is consistent with subregion duality. The resolution of the ambiguity presented in this chapter is thus exactly the one implied by the holographic correspondence.

It is worth pointing out that Uhlmann holonomy is a direct probe of entanglement. Thus, our result adds to the long-growing list of evidence that entanglement is a key ingredient in the emergence of bulk physics. Related ideas concerning entanglement and holonomy have appeared in [51, 53, 54]. However, in those papers the authors were chiefly concerned with

[2] For example, consider the case of a scalar field. We could choose to construct the subregion phase spaces in such a way that the value of this field on the common boundary is accounted for by the phase space of one subregion, and not the other, or the other way around. Alternatively, we could choose to include it in both, in the sense of the previous chapter. In any case, there is always still a consistent way to 'glue' the individual phase spaces back together. Without additional input, we don't really have any reason to pick one of these options over the others.

deformations of boundary subregions in the presence of fixed sources, whereas here we fix the boundary subregion and vary the sources. A unified picture of the results in those papers and the present one is likely to exist, and a potential approach to this will be presented in Section 4.4.1.

This chapter begins with a brief review in Section 4.2 of Uhlmann holonomy, and the related notions of fidelity and parallel purifications. In Section 4.3, we describe the construction of holographic states reduced to a subregion, and use a replica trick to find a convenient formula for the fidelity of such states. This allows us to find a sequence of parallel purifications along any given curve of such states, and to compute the Uhlmann phase of the curve. In Section 4.4, we demonstrate the equivalence between the curvature of this phase and the symplectic form of the entanglement wedge, and describe the way in which the boundary ambiguity is resolved. We also comment upon the existence of edge modes corresponding to deformations of the HRT surface. We conclude the chapter in Section 4.5 with some brief discussion and comments on possible applications.

4.2 Uhlmann holonomy and fidelity

In this section we will briefly describe and motivate Uhlmann holonomy. The proofs of several claims below can be found in the literature, e.g. in [38, 145–147].

Suppose ρ is a density matrix acting on a Hilbert space $\mathcal{H}$. A purification of ρ is a pure state $|\psi\rangle$ (which we will assume for simplicity is normalised) in an extended Hilbert space $\mathcal{H}\otimes\mathcal{H}'$ such that

$$\rho = \mathrm{tr}' \left|\psi\right\rangle \left\langle\psi\right| , \tag{4.5}$$

where tr' denotes a partial trace over $\mathcal{H}'$. The auxiliary space $\mathcal{H}'$ can be any Hilbert space one wants, and for each choice there can exist many possible purifications of a given density matrix.

Let us suppose that by measuring a system at two different times we determine that it is initially in one state ρ_1, and then subsequently in a different state ρ_2. Let us also assume also that these density matrices arise as reductions of some pure states $\left|\psi_1\right\rangle, \left|\psi_2\right\rangle$ in an extended system, but that we know nothing else about those states. Despite our ignorance about each of the purifications by themselves, we can say something about the relationship between them. In particular, the transition probability for $\left|\psi_1\right\rangle \to \left|\psi_2\right\rangle$ is

$$\left|\left\langle\psi_2|\psi_1\right\rangle\right|^2. \tag{4.6}$$

The key idea of Uhlmann is to assume that $|\psi_1\rangle\,,\,|\psi_2\rangle$ maximise this probability. If we are in a classical regime in which the transition probability distribution is sharply peaked, then on statistical grounds this assumption is a good approximation. We call purifications which satisfy this maximisation condition 'parallel'.

The following relation holds if and only if $|\psi_1\rangle\,,\,|\psi_2\rangle$ are parallel purifications:

$$|\langle\psi_2|\psi_1\rangle| = \mathrm{tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right). \tag{4.7}$$

Here the square root of a positive Hermitian operator is just defined in terms of its spectrum. The quantity on the right-hand side is known as the fidelity of ρ_1, ρ_2, and it is the square root of a generalisation of transition probability to mixed states. This result, sometimes known as Uhlmann's theorem, provides a useful criterion for determining when two purifications are parallel, and we will make use of it in this chapter. It is proven, for example, in [99].

Parallel purifications are not unique, because if $|\psi_1\rangle\,,\,|\psi_2\rangle$ are purifications satisfying (4.7), then so are

$$e^{if_1}(I\otimes U)\,|\psi_1\rangle\,,\quad e^{if_2}(I\otimes U)\,|\psi_2\rangle\,, \tag{4.8}$$

where f_1, f_2 are any two real numbers, and U is any unitary operator acting on the auxiliary Hilbert space $\mathcal{H}'$.[3] Indeed, the transition probability (4.6) is unaffected if we change the states in this way. By choosing f_1, f_2, U appropriately, one can in fact obtain all possible parallel purifications of ρ_1, ρ_2.

Suppose now that we have a closed curve C of density matrices acting on $\mathcal{H}$, and let $\rho_1, \rho_2, \ldots, \rho_n$ be a sequence of n density matrices ordered along this curve. Let us assume that we have a sequence of states $|\psi_1\rangle\,,\,|\psi_2\rangle\,,\ldots,\,|\psi_n\rangle$ in an extended Hilbert space $\mathcal{H}\otimes\mathcal{H}'$ such that each $|\psi_i\rangle$ purifies ρ_i, and such that each consecutive pair $|\psi_i\rangle\,,\,|\psi_{i+1}\rangle$ of states is parallel. Consider the limit in which $n\to\infty$ and the density matrices ρ_i densely cover the curve C. We will assume that one can choose the phases of the purifications $|\psi_i\rangle$ is such a way that they converge in this limit to a dense cover of a curve $\tilde{C}$ in $\mathcal{H}\otimes\mathcal{H}'$. Then we say that $\tilde{C}$ is a parallel lift of C.

One can directly construct parallel lifts for curves of faithful states[4] in the following way. Let

[3] The two phases e^{if_1}, e^{if_2} can be different, but U has to be the same on both states. Note that we could have set $f_1 = 0$, since we can include a phase shift in the operator U. However, we find it clearer to write things this way.

[4] Faithful states are those with an invertible density matrix. All the states we consider in this chapter are either pure or faithful.

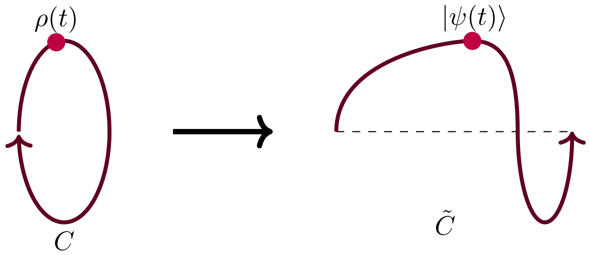


Figure 4.3: A curve C of density matrices gives rise to a parallel lift $\tilde{C}$ of purifications. Even if C is closed, it may be impossible for $\tilde{C}$ to be, because of curvature in the Uhlmann holonomy.

$t \in [0,1]$ be a parameter along C such that $\rho = \rho(t)$ is the density matrix at t, and consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \left(\int_0^\infty \mathrm{d}s\, e^{-s\rho(t)} \dot{\rho}(t) e^{-s\rho(t)} \otimes I \right) |\psi(t)\rangle \,. \tag{4.9}$$

The integral on the right-hand side is convergent because ρ is a positive operator. Along with an initial condition $|\psi(0)\rangle$ that purifies $\rho(0)$, this equation may be solved to give a curve in $\mathcal{H} \otimes \mathcal{H}'$, and it may be verified that this curve is a parallel lift of C. Of course, this is not the unique parallel lift of C, as (4.8) is still allowed, the continuous version of which is

$$|\psi(t)\rangle \to e^{if(t)} (I \otimes U) |\psi(t)\rangle \,, \tag{4.10}$$

for some real function $f : [0,1] \to \mathbb{R}$, and constant unitary U acting on $\mathcal{H}'$. If we fix the initial condition $|\psi(0)\rangle$, then one may set $U = 1, f(0) = 0$, and the space of all allowed parallel lifts of $\rho(t)$ is given by curves of the form $e^{if(t)} |\psi(t)\rangle$. In this way, parallel lifts of density matrices provide a notion of parallel transport of purifications modulo phase shifts. This is Uhlmann holonomy.

It is worth noting that although C may be a closed curve, in general its parallel lift $\tilde{C}$ is not (even up to phase shifts), as shown in Figure 4.3. This is because a purification will sometimes not return to itself upon being parallelly transported around the curve. Indeed, we are only guaranteed

$$|\psi(1)\rangle = (I \otimes X) |\psi(0)\rangle \,, \tag{4.11}$$

where X is a unitary operator acting on $\mathcal{H}'$. In other words, there is non-trivial curvature in the Uhlmann holonomy. This is due to entanglement between $\mathcal{H}$ and $\mathcal{H}'$. We cannot eliminate X by doing a transformation of the form (4.10), because the operator U must be constant, and so acts in the same way on both sides of (4.11).

Consider now the quantity γ defined by

$$e^{i\gamma} = \lim_{n\to\infty} \langle\psi_1|\psi_n\rangle \langle\psi_n|\psi_{n-1}\rangle \dots \langle\psi_3|\psi_2\rangle \langle\psi_2|\psi_1\rangle\,, \tag{4.12}$$

i.e. the Berry phase along $\tilde{C}$. This is clearly invariant under a change in parallel purifications $|\psi_i\rangle \to e^{if_i}(I\otimes U)\,|\psi_i\rangle$, and so is uniquely defined for any closed curve C of density matrices. We will refer to it as the Uhlmann phase of such a curve. For the special case of a curve of density matrices representing pure states, the Uhlmann phase reduces to the Berry phase.

Uhlmann holonomy may be viewed as a map from a curve C in the space of density matrices, and an initial purification $|\psi\rangle \in \mathcal{H}\otimes\mathcal{H}'$, to the unitary operator X in (4.11) (modulo phase shifts). Since the group of unitary operators acting on $\mathcal{H}'$ is in general non-abelian, it is clear that the Uhlmann holonomy is also non-abelian. However, in the classical regime the effects of operator ordering become subleading, and so we can expect to be able to approximate $e^{i\gamma}$ as the holonomy of an abelian $U(1)$ connection on the space of density matrices.[5] It is the curvature of this connection that will interest us the most in the next section.

4.3 Holographic Uhlmann holonomy

In this Section, we will calculate the Uhlmann phase in a holographic theory. To start, we will define the states of interest, and find an expression for their fidelity. This will then allow us to invoke Uhlmann's theorem to find parallel purifications, and from there compute the Uhlmann holonomy.

Consider a d-dimensional holographic CFT. Let us define the following class of states:

$$|\lambda\rangle = \mathrm{T}\exp\Big(-\int_{\tau<0} \mathrm{d}\tau\,\mathrm{d}^{d-1}x\,\lambda(\tau,x)\cdot\mathcal{O}(\tau,x)\Big)\,|0\rangle\,. \tag{4.13}$$

Here $\mathcal{O}$ is supposed to denote all possible single trace operators dual to bulk fields, and the parameter λ is a function which sources these fields. The T denotes a Euclidean time τ ordering[6], and the remaining coordinates x are the spatial ones. The state $|0\rangle$ on the right-hand side is usually the vacuum, whose wavefunctional is obtained by doing a Euclidean path integral over half of a d-sphere. It could also be a more complicated background state such as the thermofield double in two copies of the CFT, whose wavefunctional arises from a Euclidean path integral

[5] The reader may be concerned with the imprecision in the justification of this statement. We will only comment that, in the holographic case described in this chapter, $e^{i\gamma}$ does indeed take this form in the large N limit.

[6] This is the same time-ordering used in the path integral definition of $|0\rangle$.

over $S^{d-1} \times I_{\beta/2}$, where $I_{\beta/2}$ is an interval of length $\beta/2$ and β is the inverse temperature. Let us label the manifold on which this path integral is performed $\mathcal{M}^-$. The effect of the operator in front of $|0\rangle$ is to introduce additional sources on $\mathcal{M}^-$ in this path integral.

The conjugate states to $|\lambda\rangle$ may be written

$$\langle\lambda| = \langle 0|\,\mathrm{T}\exp\Big(-\int_{\tau>0} \mathrm{d}\tau\, \mathrm{d}^{d-1}x\, \lambda^*(-\tau, x)\cdot \mathcal{O}^\dagger(\tau, x)\Big). \tag{4.14}$$

One sees that $\langle\lambda|$ is related to $|\lambda\rangle$ by a complex conjugation and reflection of the sources across $\tau = 0$. Following [28], we will refer to this transformation as $\mathbb{Z}_2 + \mathcal{C}$, where $\mathbb{Z}_2$ refers to the time relection, and $\mathcal{C}$ refers to the complex conjugation. Let us call the reflected manifold on which this state is prepared $\mathcal{M}^+$.

The inner product of two such states may be evaluated as a path integral over the manifold obtained by gluing $\mathcal{M}^-$ and $\mathcal{M}^+$ at their boundaries. By 'gluing', we mean identifying all the fields there, and summing over them. At leading order in the classical large N limit, the holographic dictionary allows us to write this as

$$\langle\lambda_2|\lambda_1\rangle = e^{-S(\lambda_1,\lambda_2)}, \tag{4.15}$$

where $S(\lambda_1, \lambda_2)$ is the on-shell gravitational Euclidean action evaluated on a $(d+1)$-dimensional bulk with boundary conditions matching the sources $\lambda_1(\tau, x)$ for $\tau < 0$ (i.e. on $\mathcal{M}^-$), and $\lambda_2^{\mathrm{T}*}(\tau, x)$ for $\tau > 0$ (i.e. on $\mathcal{M}^+$). Here the superscript $^{\mathrm{T}}$ denotes a time reflection, so $\lambda_2^{\mathrm{T}*}(\tau, x) = \lambda_2^*(-\tau, x)$. In this chapter, unless stated otherwise, all bulk actions are Euclidean. Figure 4.4 contains an illustration of the path integrals for $|\lambda\rangle$, $\langle\lambda|$ and $\langle\lambda_2|\lambda_1\rangle$.

We will use ϕ to denote the collection of bulk fields dual to boundary operators. It was shown in [28] that, at leading order in large N, the Berry curvature for normalised states of the above form matches exactly with the symplectic form of the bulk fields, where bulk field variations $\delta\phi$ are related with changes in boundary sources $\delta\lambda$ via the holographic dictionary.

Let us now fix a proper subregion $A \subset \partial\mathcal{M}^-$ of the boundary CFT at $\tau = 0$, and factorise the boundary Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, where $\mathcal{H}_{A,\bar{A}}$ are the Hilbert spaces for the degrees of freedom in $A, \bar{A}$ respectively, and $\bar{A}$ is the complement of A. The state in A in the presence of the sources λ is given by the density matrix

$$\rho(\lambda) = e^{S(\lambda,\lambda)}\, \mathrm{tr}_{\bar{A}}\, |\lambda\rangle\, \langle\lambda|\,, \tag{4.16}$$

i.e. by tracing over all degrees of freedom in $\bar{A}$. The prefactor involving $S(\lambda, \lambda)$ is necessary for the correct normalisation. This density matrix can be prepared by computing a path integral

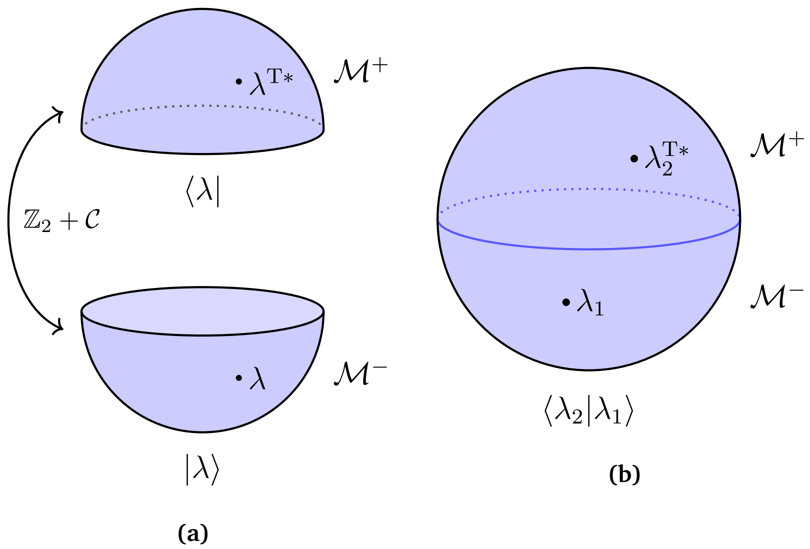


Figure 4.4: (a) The states $|\lambda\rangle$ we are considering are prepared by a Euclidean path integral. The function λ parametrises insertions of operators in this path integral. The conjugate states $\langle\lambda|$ are prepared by doing the same path integral, but with everything acted upon by $\mathbb{Z}_2+\mathcal{C}$, where $\mathbb{Z}_2$ is Euclidean time reflection and $\mathcal{C}$ is complex conjugation. **(b)** The inner product of two such states is computed by doing a path integral on the manifold obtained by gluing together the two constituent manifolds along their boundaries. At large N, this manifold sets the boundary conditions for the bulk fields.

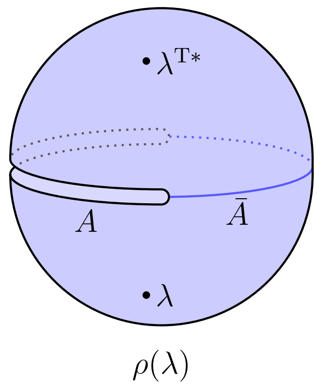


Figure 4.5: The density matrix $\rho(\lambda)$ for a subregion A may be prepared by taking the path integrals for $|\lambda\rangle$ and $\langle\lambda|$, and gluing along $\bar{A}$, the complement of A.

over the manifold obtained by gluing $\bar{A} \subset \partial\mathcal{M}^-$ to its mirror image under $\mathbb{Z}_2 + \mathcal{C}$ in $\partial\mathcal{M}^+$. This is shown in Figure 4.5. The state $\rho(\lambda)$ is in general mixed due to the presence of entanglement between $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$ in $|\lambda\rangle$.

4.3.1 Fidelity from a replica trick

Suppose we have prepared two such states $\rho_1 = \rho(\lambda_1), \rho_2 = \rho(\lambda_2)$ in this way. In this section we will find an expression for the fidelity of these two states. The fidelity of holographic states has previously been considered in [7, 19, 121] and others.

Consider the operator $\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}$, whose trace is the fidelity of ρ_1 and ρ_2. This operator is positive since

$$\sqrt{\rho_1}\rho_2\sqrt{\rho_1} = (\sqrt{\rho_2}\sqrt{\rho_1})^\dagger\sqrt{\rho_2}\sqrt{\rho_1}. \tag{4.17}$$

Furthermore, by (4.7) the fidelity is equal to the inner product of two normalised states, and so the trace of this operator is less than or equal to 1. Note that, for typical states in a QFT, reduced states in proper subregions are faithful, so we can conclude that the operator is invertible and that all its eigenvalues lie strictly between 0 and 1.

Let us define the 'replicated fidelity'[7]

$$F_k = \mathrm{tr}\Big((\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^k\Big). \tag{4.19}$$

By the above considerations, F_k is analytic in k, and absolutely bounded by 1 for $\mathrm{Re}\, k \geq \frac{1}{2}$. By Carlson's theorem, F_k is therefore uniquely determined in this range by the values it takes on positive integers k. Our strategy to find the fidelity will be to compute F_k for $k \in \mathbb{Z}_{>0}$, and then to analytically continue back to $k = \frac{1}{2}$. This is easier than a direct calculation of the fidelity because the cyclic property of the trace means we can write

$$F_k = \mathrm{tr}\Big((\rho_1\rho_2)^k\Big) \qquad \text{for } k \in \mathbb{Z}_{>0}. \tag{4.20}$$

For the states we are considering, one may compute (4.20) with a path integral. The manifold over which this path integral should be evaluated contains $2k$ copies of each of $\mathcal{M}^-$ and $\mathcal{M}^+$,

[7] It is worth pointing out that this replica trick is different to the one employed in [109]. In that paper, the 'relative Rényi entropy'

$$S_k = \frac{1}{k-1}\log\mathrm{tr}\left(\left(\rho_1^{\frac{1-k}{2k}}\rho_2\rho_2^{\frac{1-k}{2k}}\right)^k\right) \tag{4.18}$$

was the object considered. This is supposed to give the fidelity in the limit $k \to \frac{1}{2}$. It would be interesting to see if one could use the relative Rényi entropy to get similar results to the ones found here.

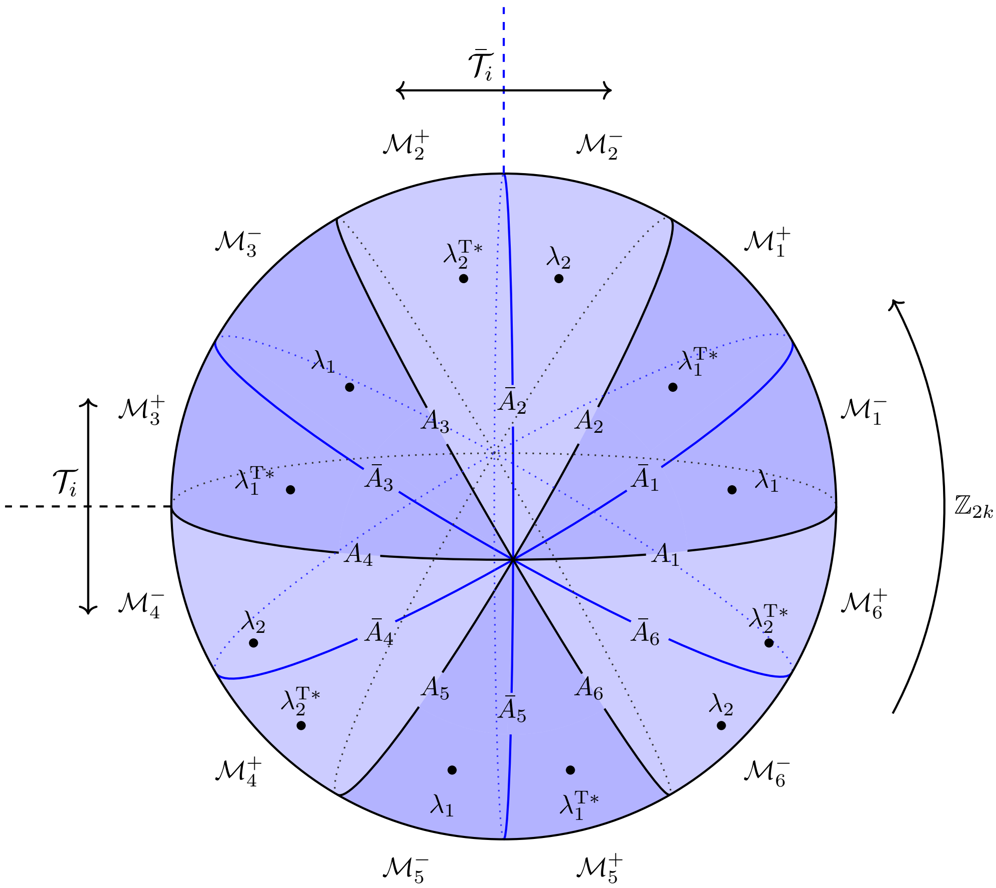


Figure 4.6: The $2k$-replicated manifold on which the path integral for the replicated fidelity F_k is performed. The arrows show the actions of $\mathbb{Z}_{2k}$ replica symmetry, and the two types of $\mathbb{Z}_2 + \mathcal{C}$ symmetry called $\mathcal{T}_i$ and $\bar{\mathcal{T}}_i$. (The case shown is for $k = 3$.)

which we label $\mathcal{M}_i^-, \mathcal{M}_i^+$ respectively, with $i = 1, \ldots, 2k$ (we will use notation in which the index i is taken mod $2k$). The subregions A and $\bar{A}$ will be labelled $A_i, \bar{A}_i$ respectively in $\partial\mathcal{M}_i^-$, and we will temporarily use $A_i^+, \bar{A}_i^+$ to label their mirror images in $\partial\mathcal{M}_i^+$. In the path integral for (4.20), one glues $\bar{A}_i$ to $\bar{A}_i^+$, and A_i to A_{i+1}^+. One then inserts sources λ_1, λ_2 on $\mathcal{M}_i^-$, and $\lambda_1^{\mathrm{T}*}, \lambda_2^{\mathrm{T}*}$ on $\mathcal{M}_i^+$, for (w.l.o.g.) odd/even i respectively. In this way one constructs a path integral on a $2k$-fold replicated version of the original manifold, which is portrayed in Figure 4.6.

At large N, we may use the holographic dictionary to write the replica path integral in terms of the bulk action. Let $S^{(k)}(\lambda_1, \lambda_2)$ be the gravitational action evaluated on the on-shell bulk field configuration ϕ whose boundary conditions are set by the sources in the replica path integral as described above. Then at leading order in N we have

$$F_k = \exp\Big(kS(\lambda_1, \lambda_1) + kS(\lambda_2, \lambda_2) - S^{(k)}(\lambda_1, \lambda_2)\Big), \tag{4.21}$$

where the contributions of $S(\lambda_1, \lambda_1)$ and $S(\lambda_2, \lambda_2)$ come from the normalisation in (4.16).

When $\lambda_1 = \lambda_2$, the replica path integral has the following symmetries:

- One may cyclically permute the individual replicas, sending $\mathcal{M}_i^- \to \mathcal{M}_{i+1}^-$ and $\mathcal{M}_i^+ \to \mathcal{M}_{i+1}^+$. This $\mathbb{Z}_{2k}$ symmetry is called replica symmetry.

- One may reflect the entire replicated manifold across $A_i \cup A_{i+k}$ (i.e. the black circles in Figure 4.6), while also taking the complex conjugate of all the sources. This is a version of the original $\mathbb{Z}_2 + \mathcal{C}$ symmetry. We refer to this as $\mathcal{T}_i$ symmetry.

- Similarly, one may reflect the entire replicated manifold across $\bar{A}_i \cup \bar{A}_{i+k}$ (i.e. the blue circles in Figure 4.6), while also taking the complex conjugate of all the sources. This is another version of $\mathbb{Z}_2 + \mathcal{C}$ symmetry, and we refer to it as $\bar{\mathcal{T}}_i$ symmetry.

However, when $\lambda_1 \neq \lambda_2$, some of these symmetries are violated:

- Replica symmetry is broken down to the subgroup $\mathbb{Z}_k$ generated by the permutation sending $\mathcal{M}_i^- \to \mathcal{M}_{i+2}^-$ and $\mathcal{M}_i^+ \to \mathcal{M}_{i+2}^+$.

- $\mathcal{T}_i$ symmetry is broken entirely.

- However, $\bar{\mathcal{T}}_i$ symmetry is maintained.

We will assume that the symmetries that hold on the boundary continue to hold in the bulk, i.e. they are not spontaneously broken.

From now on, we will write $\lambda_2 = \lambda_1 + \delta_1\lambda$, and assume that $\delta_1\lambda$ is small. Let us use ϕ_1 to denote the bulk fields at $\delta_1\lambda = 0$. We can decompose this bulk in the following way. Let Υ be the codimension 2 surface in the bulk which is fixed by replica symmetry. Also, let $\Sigma_i, \bar{\Sigma}_i$ be codimension 1 surfaces extending from Υ to $A_i, \bar{A}_i$ respectively, such that $\mathcal{T}_i$ fixes Σ_i, $\bar{\mathcal{T}}_i$ fixes $\bar{\Sigma}_i$, and replica symmetry maps $\Sigma_i \to \Sigma_{i+1}$, $\bar{\Sigma}_i \to \bar{\Sigma}_{i+1}$. We can then divide the bulk into $4k$ pieces $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$, where $\mathcal{N}_i^-$ is bounded by $\Sigma_i \cup \bar{\Sigma}_i \cup \mathcal{M}_i^-$, and $\mathcal{N}_i^+$ is bounded by $\bar{\Sigma}_i \cup \Sigma_{i+1} \cup \mathcal{M}_i^+$. This is depicted in Figure 4.7, which may be thought of as the cross section of Figure 4.6 when cut along the plane of the page.

When $\delta_1\lambda$ is allowed to become non-zero, the bulk field configuration picks up a perturbation $\delta_{1,2}\phi$ obeying the linearised equations of motion, and consistent with the $\mathbb{Z}_k$ subgroup of replica

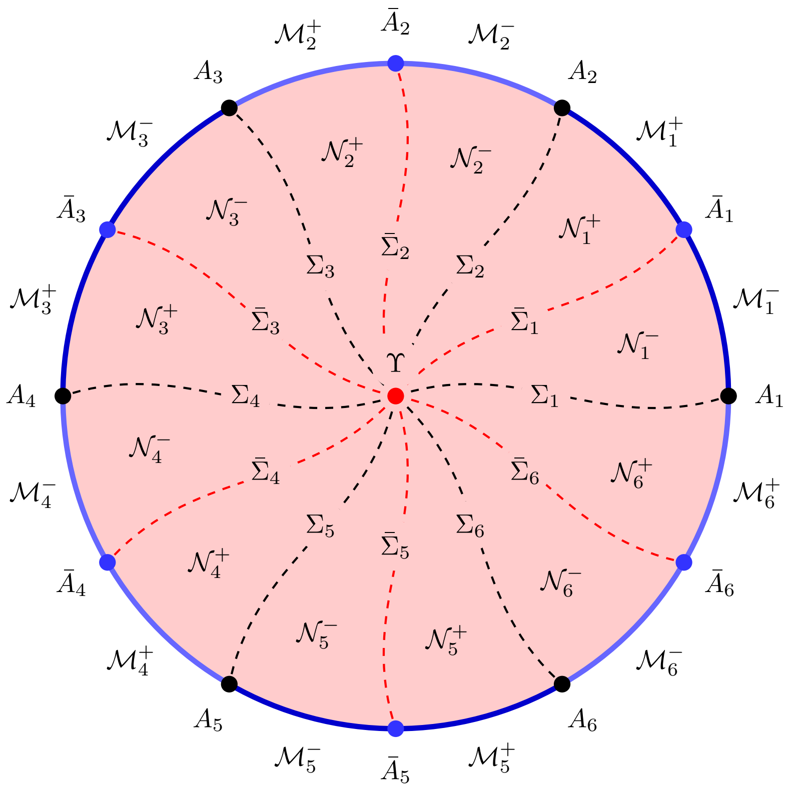


Figure 4.7: The bulk corresponding to F_k. Assuming the symmetries of the boundary continue to hold in the bulk, we may decompose it as shown. (The case shown is for $k = 3$.)

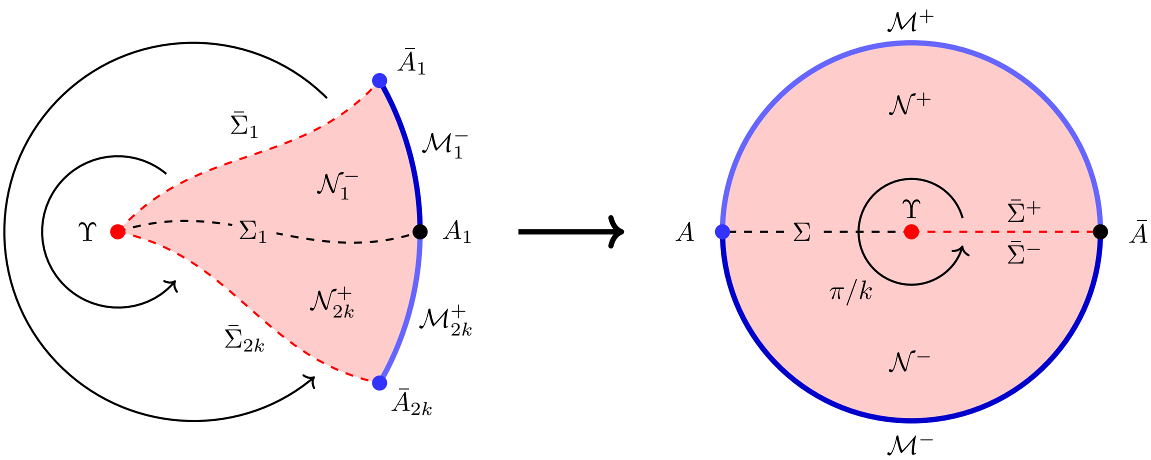


Figure 4.8: $S^{(k)}$ may be understood in terms of the on-shell action for a bulk manifold $\mathcal{N}$ with a conical defect of opening angle π/k at Υ. This manifold is constructed by identifying $\bar{\Sigma}^- = \bar{\Sigma}_1$ with $\bar{\Sigma}^+ = \bar{\Sigma}_{2k}$ in $\mathcal{N}_1^- \cup \mathcal{N}_{2k}^+$.

symmetry, and $\bar{\mathcal{T}}_i$ symmetry. In terms of the Lagrangian density $L = L[\phi_1 + \delta_{1,2}\phi]$, we have

$$\begin{aligned} S^{(k)}(\lambda_1, \lambda_1 + \delta_1\lambda) &= \sum_{i=1}^{2k} \left(\int_{\mathcal{N}_i^-} L + \int_{\mathcal{N}_i^+} L \right) \\ &= k\left(\int_{\mathcal{N}_1^-} L + \int_{\mathcal{N}_1^+} L + \int_{\mathcal{N}_{2k}^-} L + \int_{\mathcal{N}_{2k}^+} L \right) \\ &= 2k\,\mathrm{Re}\left(\int_{\mathcal{N}_1^-} L + \int_{\mathcal{N}_{2k}^+} L \right), \end{aligned} \tag{4.22}$$

where the second line follows from $\mathbb{Z}_k$ replica symmetry, while the third line follows from $\bar{\mathcal{T}}_1$ and $\bar{\mathcal{T}}_{2k}$ symmetry. Thus, we can understand $S^{(k)}(\lambda_1, \lambda_2)$ purely in terms of the contribution to the action from $\mathcal{N} = \mathcal{N}_1^- \cup \mathcal{N}_{2k}^+$.

At $\delta_1\lambda = 0$, by replica symmetry the fields at $\bar{\Sigma}_1$ are equal to those at $\bar{\Sigma}_{2k}$, so we can smoothly identify these two boundaries of $\mathcal{N}$. The result is a bulk manifold with the same boundary conditions as the density matrix $\rho(\lambda_1)$, and which is smooth everywhere except for a conical singularity of opening angle π/k at Υ. This is demonstrated in Figure 4.8. As $k \to \frac{1}{2}$, this conical singularity goes away, and we are left with just the action evaluated on the on-shell bulk field configuration matching the boundary conditions of $\rho(\lambda_1)$. Thus, upon analytic continuation to $k = \frac{1}{2}$, we may write

$$S^{(k)}(\lambda_1, \lambda_1) \to S(\lambda_1, \lambda_1). \tag{4.23}$$

When $\delta\lambda$ is small but non-zero, we can no longer use replica symmetry to compare the fields at $\bar{\Sigma}_1$ and $\bar{\Sigma}_{2k}$. However, we can still treat the perturbation $\delta_{1,2}\phi$ as living on $\mathcal{N}$. Let us simplify the notation by discarding subscripts when referring to parts of this spacetime, so writing $\Sigma = \Sigma_1$, $\bar{\Sigma} = \bar{\Sigma}_1 = \bar{\Sigma}_{2k}$, and so on. Note that the fields at $\bar{\Sigma}_1$ differ from those at $\bar{\Sigma}_{2k}$, and so $\delta_{1,2}\phi$ must be discontinuous when crossing $\bar{\Sigma}$. When we need to refer to the field on either side of $\bar{\Sigma}$, we will use the notation $\bar{\Sigma}^- = \bar{\Sigma}_1$ and $\bar{\Sigma}^+ = \bar{\Sigma}_{2k}$. Besides the discontinuity at $\bar{\Sigma}$, and the conical defect at Υ, $\delta_{1,2}\phi$ is otherwise smooth on $\mathcal{N}$.

We need to characterise the discontinuity at $\bar{\Sigma}$ in such a way that permits an easy analytic continuation in k. To do so, let $\delta_{\overset{\frown}{1,2}}\phi$ be the bulk field variation obtained by acting on $\delta_{1,2}\phi$ once with $\mathbb{Z}_{2k}$ replica symmetry, and let

$$\delta_1\phi = \delta_{1,2}\phi + \delta_{\overset{\frown}{1,2}}\phi, \tag{4.24}$$

$$\tilde{\delta}_1\phi = \delta_{1,2}\phi - \delta_{\overset{\frown}{1,2}}\phi. \tag{4.25}$$

By linearity, $\delta_1\phi, \tilde{\delta}_1\phi$ are solutions to the linearised equations of motion. Under the action of $\mathbb{Z}_{2k}$ replica symmetry, $\delta_1\phi, \tilde{\delta}_1\phi$ change by a $\pm$ sign respectively. From this we may deduce the following about $\delta_1\phi, \tilde{\delta}_1\phi$ restricted to $\mathcal{N}$:

- $\delta_1\phi$ is continuous at $\bar{\Sigma}$, while $\tilde{\delta}_1\phi$ changes sign when crossing $\bar{\Sigma}$.
- The boundary conditions for $\delta_1\phi$ are given by $\delta\lambda^{\mathrm{T}*}$ on $\mathcal{M}^+$ and $\delta\lambda$ on $\mathcal{M}^-$.
- The boundary conditions for $\tilde{\delta}_1\phi$ are given by $\delta\lambda^{\mathrm{T}*}$ on $\mathcal{M}^+$ and $-\delta\lambda$ on $\mathcal{M}^-$.

These three conditions, along with $\bar{\mathcal{T}}_i$ symmetry, are sufficient to determine $\delta_1\phi, \tilde{\delta}_1\phi$ in the entire bulk replicated manifold, and so must be sufficient to determine $\delta_1\phi, \tilde{\delta}_1\phi$ in $\mathcal{N}$. They are simple to understand for analytically continued k, and one may recover $\delta_{1,2}\phi$ from $\delta_{1,2}\phi = \frac{1}{2}(\delta_1\phi + \tilde{\delta}_1\phi)$.

When $k \to \frac{1}{2}$, the conical defect at Υ vanishes. However, Υ remains important, because it is the boundary of $\bar{\Sigma}$, which is where $\tilde{\delta}_1\phi$ is discontinuous. For the usual reasons,[8] at $k = \frac{1}{2}$, Υ coincides with the surface of minimal area which is homologous to A, i.e. the HRT surface.

Also, it should be clear that at $k = \frac{1}{2}$ we may write $\phi_2 = \phi_1 + \delta_1\phi$, where ϕ_2 is the bulk field configuration for the boundary conditions λ_2 at $\mathcal{M}^-$ and $\lambda_2^{\mathrm{T}*}$ at $\mathcal{M}^+$. This is because the conditions on $\delta_1\phi$ listed above exactly agree with the conditions on $\phi_2 - \phi_1$.

Let us briefly comment on the symmetries of $\mathcal{N}$.

- There is a $\mathbb{Z}_2 + \mathcal{C}$ symmetry which reflects everything across $\Sigma \cup \bar{\Sigma}$, including swapping $\bar{\Sigma}^-$ and $\bar{\Sigma}^+$, and complex conjugates the fields. This is in some sense inherited from the $\mathcal{T}_i$ symmetry of the replicated spacetime. Under this symmetry, ϕ_1 and $\delta_1\phi$ are invariant, but $\tilde{\delta}\phi \to -\tilde{\delta}\phi$.
- There is another type of $\mathbb{Z}_2 + \mathcal{C}$ symmetry, distinct from the first, which acts only on the fields at $\bar{\Sigma}^{\pm}$. It reflects all components of the fields in time, and complex conjugates them, but it does *not* swap $\bar{\Sigma}^-$ and $\bar{\Sigma}^+$. This symmetry is inherited from the $\bar{\mathcal{T}}_i$ symmetry of the replicated spacetime. All of the fields $\phi, \delta_1\phi, \tilde{\delta}_1\phi$ are invariant under this symmetry.

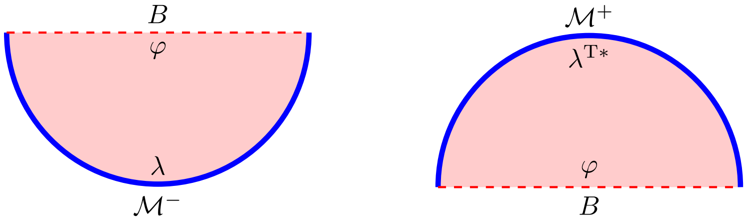
We may now analytically continue (4.22) to $k = \frac{1}{2}$, and write

$$S^{(k)}(\lambda_1, \lambda_1 + \delta\lambda) = \mathrm{Re}(S[\phi_{1,2}]), \tag{4.26}$$

where $S[\phi_{1,2}]$ denotes the action evaluated for the field configuration $\phi_{1,2} = \phi_1 + \frac{1}{2}(\delta_1\phi + \tilde{\delta}_1\phi)$ on $\mathcal{N}$. Therefore, using (4.21) we may write the fidelity as

$$\mathrm{tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right) = F_{\frac{1}{2}} = \exp\left(\frac{1}{2}S(\lambda_1, \lambda_1) + \frac{1}{2}S(\lambda_2, \lambda_2) - \mathrm{Re}(S[\phi_{1,2}])\right). \tag{4.27}$$

[8] The main reference is [112]. Very briefly, one may include a term in the action proportional to the area of Υ, in order to allow for the conical singularity. In a saddlepoint approximation this area is minimised, and this effect persists in the limit $k \to \frac{1}{2}$.



(a) **(b)**

Figure 4.9: The boundary conditions for the bulk wavefunctionals **(a)** $\langle\varphi|\lambda\rangle$ and **(b)** $\langle\lambda|\varphi\rangle$.

Thus, we have found an expression for the fidelity of the two holographic states ρ_1, ρ_2 in terms of the action for the bulk field configuration $\phi_{1,2}$.

4.3.2 Parallel purifications from Uhlmann's theorem

It is the objective of this section to construct parallel purifications $|\psi_1\rangle\,, |\psi_2\rangle$ of the two density matrices $\rho(\lambda_1), \rho(\lambda_2)$. This will be done by using the expression for the holographic fidelity (4.27) in Uhlmann's theorem (4.7).

For the first state we will simply take the normalised version of the state $|\lambda_1\rangle$ constructed with (4.13):

$$|\psi_1\rangle = e^{\frac{1}{2}S(\lambda_1,\lambda_1)}\,|\lambda_1\rangle\,. \tag{4.28}$$

This is a purification of $\rho(\lambda_1)$ by construction.

We will use the bulk theory to construct $|\psi_2\rangle$, so let us start by recalling some facts. At large N, the wavefunctional for the bulk fields in the state $|\lambda\rangle$ may be computed with a bulk path integral. In particular, one may write

$$\langle\varphi|\lambda\rangle = \int_\lambda^\varphi \mathrm{D}\phi\, e^{-S^-[\phi]}, \tag{4.29}$$

where the integral is done over all bulk field configurations whose boundary conditions are set by λ at the asymptotic boundary $\mathcal{M}^-$, and φ on a bulk surface B, as shown in Figure 4.9a. We should emphasise our notation here: ϕ denotes the bulk fields, whereas φ denotes the boundary data at B. $S^-[\phi]$ is the bulk action of the field configuration with these boundary conditions.

The wavefunctional of the conjugate states $\langle\lambda|$ may be written

$$\langle\lambda|\varphi\rangle = \int_\varphi^\lambda \mathrm{D}\phi\, e^{-S^+[\phi]} \tag{4.30}$$

where the integral is done over all bulk field configurations with boundary data $\lambda^{\mathrm{T}*}$ at $\mathcal{M}^+$, and φ at B. This is shown in Figure 4.9b. $S^+[\phi]$ is the bulk action for these field configurations.

The states $|\varphi\rangle$ make up a normalised basis for the bulk Hilbert space. The identity in this basis is

$$\int \mathrm{D}\varphi \, |\varphi\rangle \, \langle\varphi| \, . \tag{4.31}$$

This implies that

$$\begin{aligned}
\langle\lambda_2|\lambda_1\rangle &= \int \mathrm{D}\varphi \, \langle\lambda_2|\varphi\rangle \, \langle\varphi|\lambda_1\rangle \\
&= \int \mathrm{D}\varphi \left(\int_{\varphi}^{\lambda_2} \mathrm{D}\phi_+ \, e^{-S_+[\phi_+]}\right)\left(\int_{\lambda_1}^{\varphi} \mathrm{D}\phi_- \, e^{-S^-[\phi_-]}\right) \\
&= \int_{\lambda_1}^{\lambda_2} \mathrm{D}\phi \, e^{-S[\phi]}.
\end{aligned} \tag{4.32}$$

Where the last integral is done over all bulk field configurations whose boundary conditions are given by λ_1 at $\mathcal{M}^-$, and $\lambda_2^{\mathrm{T}*}$ at $\mathcal{M}^+$. The action in the last line is $S[\phi] = S^+[\phi_+] + S^-[\phi_-]$. At large N, a stationary phase approximation recovers (4.15).

One may determine what kind of boundary conditions are set by the state $|\varphi\rangle$ by looking at the variation of the bulk action $S^- = \int_{\text{bulk}} L$. When the bulk equations of motion are obeyed, one has

$$\delta S^- = \int_{\mathcal{M}^-} \theta[\phi, \delta\phi] + \int_B \theta[\phi, \delta\phi], \tag{4.33}$$

The contribution at B determines the form of φ by the requirement that it vanishes when φ is kept fixed. In particular this means that $\int_B \theta$ can only depend on $\delta\phi$ through $\delta\varphi$. For the toy example of a scalar field with $L = -\frac{1}{2}\,\mathrm{d}\phi \wedge * \,\mathrm{d}\phi$, we have $\theta = -\delta\phi * \mathrm{d}\phi$, so φ is just the initial data for the scalar field on B.

There is an easy way to carry out a change of basis in the bulk Hilbert space: one simply adds a boundary term of the form

$$S_B = \int_B D[\phi] \tag{4.34}$$

to the action in (4.29). One must simultaneously subtract S_B from the action in (4.30). The variation of the new action $S^- = \int_{\text{bulk}} L + \int_B D$ reads

$$\delta S^- = \int_{\mathcal{M}^-} \theta[\phi, \delta\phi] + \int_B \Big(\theta[\phi, \delta\phi] + \delta D[\phi]\Big). \tag{4.35}$$

This modifies the way in which the term at B depends on $\delta\phi$, and therefore changes the type of boundary data φ specified by the bulk state $|\varphi\rangle$. For the example of the scalar field, one might

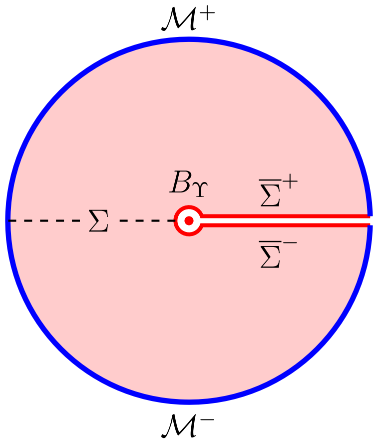


Figure 4.10: Variations of the action in the presence of discontinuities at $\bar{\Sigma}$ lead to new boundary terms at $\bar{\Sigma}^- \cup \bar{\Sigma}^+ \cup B_\Upsilon$.

pick $D = \phi * \mathrm{d}\phi$. Then we would have $\theta + \delta D = \phi\,\delta(*\,\mathrm{d}\phi)$, and hence φ would be the normal derivative of the scalar field at B, i.e. its conjugate momentum in a canonical treatment.

We should note that the boundary data φ must be invariant under $\mathbb{Z}_2 + \mathcal{C}$. This is to ensure that a Wick rotation to a real Lorentzian spacetime exists, and states without this property are not part of the bulk Hilbert space. It is also important that this does *not* mean that the dominant field configuration in any stationary phase approximation must be $\mathbb{Z}_2 + \mathcal{C}$ invariant at B. This is because stationary phase methods involve a complex deformation of the field contour. For similar reasons, S_B must be imaginary for all $\mathbb{Z}_2 + \mathcal{C}$ invariant field configurations. This ensures, for example, that the form of the identity (4.31) is the same for different possible choices of S_B.

Let us identify B with $\Sigma \cup \bar{\Sigma}$, as defined in Section 4.3.1. Let ϕ_1 be the bulk on-shell field configuration whose boundary conditions match those of $\rho(\lambda_1)$, and let $\delta_1\phi, \tilde{\delta}_1\phi$ be the field variations such that $\phi_2 = \phi_1 + \delta_1\phi$, and $\phi_{1,2} = \phi_1 + \frac{1}{2}(\delta_1\phi + \tilde{\delta}\phi)$ is the configuration relevant to the fidelity of $\rho(\lambda_1), \rho(\lambda_2)$, as discussed in the previous subsection.

Consider the classical phase space for the bulk field theory. We can view points in this phase space as describing the fields at B. Consider a map $Z_{1,2}$ from phase space to itself, whose action does not affect the fields at Σ, but maps $\phi_{1,2}|_{\bar{\Sigma}^-}$ to $\phi_{1,2}|_{\bar{\Sigma}^+}$ at $\bar{\Sigma}$ (these differ by $\tilde{\delta}_1\phi|_{\bar{\Sigma}^-}$). This defines the action of $Z_{1,2}$ at certain points in phase space, and we may extend $Z_{1,2}$ to a symplectomorphism of the full classical phase space (a.k.a. a canonical transformation of the classical variables). In the quantum theory there is a corresponding unitary operator $X_{1,2}$ which implements the action of $Z_{1,2}$. Let $\varphi^{\pm}_{1,2}$ be the boundary data for $\phi_{1,2}$ on $\Sigma \cup \bar{\Sigma}^{\pm}$ respectively. It is clear that

$$X_{1,2}\,|\varphi^-_{1,2}\rangle = e^{ix_{1,2}}\,|\varphi^+_{1,2}\rangle\,, \tag{4.36}$$

where $x_{1,2}$ is some real number that we will leave undetermined.

Let us see what happens when we insert this operator in between $\langle\lambda_2|$ and $|\lambda_1\rangle$. Using path integral notation, we have

$$\langle\lambda_2|X_{1,2}|\lambda_1\rangle = \int_{\lambda_1}^{\lambda_2} \mathrm{D}\phi_+\,\mathrm{D}\phi_-\;e^{-S^+[\phi_+]}X_{1,2}e^{-S^-[\phi_-]}. \tag{4.37}$$

The presence of $X_{1,2}$ means that the integral is done over field configurations with the property that the fields ϕ_+ at B are related to the fields ϕ_- at B by an action of $Z_{1,2}$. Also, the boundary conditions $\lambda_1, \lambda_2^{\mathrm{T}*}$ at $\mathcal{M}^-, \mathcal{M}^+$ must continue to hold.

An important question to ask is whether we can still use a stationary phase approximation to evaluate this integral. A variation of the field configurations $\phi_1, \phi_2 \to \phi_1 + \delta\phi_2, \phi_2 + \delta\phi_2$ in the integrand leads to an expression of the form

$$e^{-S[\phi_2]+\int E[\phi_2]\cdot\delta\phi_2}e^{-\int_B \theta[\phi_2,\delta\phi_2]}X_{1,2}e^{\int_B \theta[\phi_1,\delta\phi_1]}e^{-S[\phi_1]+\int E[\phi_1]\cdot\delta\phi_1}. \tag{4.38}$$

So in addition to the equations of motion, we pick up some terms at B, which could potentially be an issue. However, we should view $e^{-\int_B \theta[\phi_2,\delta\phi_2]}$ as a bulk operator, and when commuted past $X_{1,2}$, the fields in this operator are acted upon by $Z_{1,2}$. Thus,

$$e^{-\int_B \theta[\phi_2,\delta\phi_2]}X_{1,2}e^{\int_B \theta[\phi_1,\delta\phi_1]} = X_{1,2}e^{-\int_B \theta[\phi_1,\delta\phi_1]}e^{\int_B \theta[\phi_1,\delta\phi_1]} = X_{1,2}, \tag{4.39}$$

and so (4.38) becomes

$$e^{-S[\phi_2]+\int E[\phi_2]\cdot\delta\phi_2}X_{1,2}e^{-S[\phi_1]+\int E[\phi_1]\cdot\delta\phi_1}, \tag{4.40}$$

i.e. only the equations of motion remain. Thus, the stationary phase method still works. In particular, the path integral is dominated by field configurations which pick up an action of $Z_{1,2}$ when crossing B, and which obey the equations of motion elsewhere.

It seems like the dominant field configuration should be exactly $\phi_{1,2}$. However, there is one additional constraint that must be satisfied. Consider the change in the bulk action under an on-shell variation of the bulk fields $\phi_{1,2} \to \phi_{1,2} + \delta\phi$. Because of the discontinuities at $\bar{\Sigma}$, we should evaluate this action on a spacetime with $\bar{\Sigma}$ removed, and this introduces new boundaries $\bar{\Sigma}^- \cup \bar{\Sigma}^+ \cup B_\Upsilon$, as shown in Figure 4.10. The surfaces $\bar{\Sigma}^-, \bar{\Sigma}^+$ are just on either side of $\bar{\Sigma}$, while B_Υ wraps around Υ. Assuming that the holographic boundary conditions λ_1, λ_2 are fixed, the variation of the action reduces to the boundary terms

$$\delta S[\phi_{1,2}] = \int_{\bar{\Sigma}^-} \theta[\phi_{1,2},\delta\phi] - \int_{\bar{\Sigma}^+} \theta[\phi_{1,2},\delta\phi] + \int_{B_\Upsilon} \theta[\phi_{1,2},\delta\phi]. \tag{4.41}$$

The terms at $\bar{\Sigma}^-, \bar{\Sigma}^+$ do not concern us because the presence of $X_{1,2}$ means they cancel in the path integral, as just explained. However, the contribution at B_Υ is in general non-vanishing, and interacts with $X_{1,2}$ in a complicated way.

Recall from the Introduction that there is an ambiguity in the definition of θ. In particular, we are free to carry out the change $\theta[\phi, \delta\phi] \to \theta[\phi, \delta\phi] + \mathrm{d}K[\phi, \delta\phi]$. We will use this change to remove the term at B_Υ in (4.41). In other words, we pick a K such that

$$\int_{B_\Upsilon} \theta[\phi_{1,2}, \delta\phi] = -\int_{\partial B_\Upsilon} K[\phi_{1,2}, \delta\phi]. \tag{4.42}$$

Note that each of these integrals should be considered in a limit in which B_Υ tightly encloses Υ, so the left-hand side really only depends on the fields at Υ, and the right-hand side becomes an integral at Υ. Thus, it is possible to choose a $K[\phi, \delta\phi]$ satisfying the above in such a way that it only depends on the field configuration locally. From now on we will assume that we have done this transformation $\theta \to \theta + \mathrm{d}K$, so that the boundary term

$$\int_{B_\Upsilon} \theta[\phi_{1,2}, \delta\phi] \tag{4.43}$$

vanishes. This will be crucial for the resolution of the boundary ambiguity in the covariant phase space formalism, and will be discussed further in Section 4.4.2.

We should note that this ambiguity in θ does not really have any impact on the construction of the theory, and doesn't have any physical consequences. This is because all of the quantities we have discussed up to this point are integrals of θ which do not change when we do $\theta \to \theta + \mathrm{d}K$. So really, this is not an ambiguity, but rather a choice that we are free to make. The choice that we have just made is merely a convenient one that allows us to better understand the contribution of the B_Υ term in (4.41).

Note that $X_{1,2}$ commutes with observables on Σ. Bulk reconstruction [8, 59] implies that $X_{1,2}$ can therefore be treated as a unitary operator in the boundary theory acting on $\mathcal{H}_{\bar{A}}$. By this we mean that one may write $X_{1,2} = I_A \otimes X_{\bar{A}}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, where I_A is the identity acting on $\mathcal{H}_A$, and $X_{\bar{A}}$ is a unitary operator acting on $\mathcal{H}_{\bar{A}}$.[9]

We are now ready to construct the state $|\psi_2\rangle$. It is given by

$$|\psi_2\rangle = e^{\frac{1}{2}S(\lambda_2,\lambda_2)} X_{1,2}^\dagger \, |\lambda_2\rangle \, . \tag{4.44}$$

[9] Technically [8, 59] only give us a way to construct the action of this operator on a 'code subspace' of the theory. Roughly speaking, the code subspace consists of states with well-defined classical bulk duals. In our case, this is fine, because we are only ever considering the action of $X_{1,2}$ on such states.

Since $X_{1,2}$ acts on $\mathcal{H}_{\bar{A}}$, this is a genuine purification of $\rho(\lambda_2)$. It remains to show that $|\psi_1\rangle$ and $|\psi_2\rangle$ are parallel. To do this, note that the stationary phase approximation allows us to write

$$\langle\lambda_2|X_{1,2}|\lambda_1\rangle = e^{-S[\phi_{1,2}]}\,\langle\varphi^+_{1,2}|X_{1,2}|\varphi^-_{1,2}\rangle\,. \tag{4.45}$$

Using (4.36), the second factor on the right-hand side is equal to $e^{ix_{1,2}}$. Including the normalising factors in (4.28) and (4.44), and taking the absolute value, one finds

$$|\langle\psi_2|\psi_1\rangle| = \exp\biggl(\frac{1}{2}S(\lambda_1,\lambda_1) + \frac{1}{2}S(\lambda_2,\lambda_2) - \operatorname{Re} S[\phi_{1,2}]\biggr). \tag{4.46}$$

This matches exactly with the holographic fidelity found in the previous section. Therefore, by Uhlmann's theorem, $|\psi_1\rangle$ and $|\psi_2\rangle$ are parallel.

4.3.3 Uhlmann phase

By using the results of the previous subsection, we will now compute the Uhlmann phase associated with a closed curve of holographic density matrices $\rho(\lambda)$ reduced to a fixed subregion A arising from a closed curve $\lambda(t)$ of boundary sources.

First let us state our notation. Let λ_i, $i = 1,\dots,n$ be a sequence of points ordered along the curve of sources, and let $\rho_i = \rho(\lambda_i)$ be the density matrices for these sources, obtained by reducing the pure states $|\lambda_i\rangle$ (as prepared using (4.13)) to $\mathcal{H}_A$:

$$\rho_i = \operatorname{tr}_{\bar{A}}|\lambda_i\rangle\langle\lambda_i|\,. \tag{4.47}$$

Let ϕ_i be the bulk field configuration matching the boundary conditions set by the boundary sources λ_i, and let $\delta_i\phi$ be defined by $\phi_{i+1} = \phi_i + \delta_i\phi$. Furthermore, let $\phi_{i,i+1} = \phi_i + \frac{1}{2}(\delta_i\phi + \tilde{\delta}_i\phi)$ be the bulk field configuration relevant to the fidelity of $\rho(\lambda_i)$ and $\rho(\lambda_{i+1})$. We may construct symplectomorphisms $Z_{i,i+1}$ of the bulk phase space that map $\phi_{i,i+1}|_{\bar{\Sigma}^-}$ to $\phi_{i,i+1}|_{\bar{\Sigma}^+}$, and associated unitary operators $X_{i,i+1}$. Let us define the real numbers $x_{i,i+1}$ with

$$X_{i,i+1}\left|\varphi^-_{i,i+1}\right\rangle = e^{ix_{i,i+1}}\left|\varphi^+_{i,i+1}\right\rangle, \tag{4.48}$$

where $\varphi^{\pm}_{i,i+1}$ is the boundary data for $\phi_{i,i+1}$ at $\Sigma\cup\bar{\Sigma}^{\pm}$ respectively.

Consider the following sequence of states:

$$|\psi_1\rangle = e^{\frac{1}{2}S(\lambda_1,\lambda_1)}\,|\lambda_1\rangle\,, \tag{4.49}$$

$$\dots$$

$$|\psi_i\rangle = e^{\frac{1}{2}S(\lambda_i,\lambda_i)}X^\dagger_{1,2}X^\dagger_{2,3}\dots X^\dagger_{i-1,i}\,|\lambda_i\rangle \tag{4.50}$$

$$\dots$$

$$|\psi_n\rangle = e^{\frac{1}{2}S(\lambda_n,\lambda_n)}X^\dagger_{1,2}X^\dagger_{2,3}\dots X^\dagger_{n-1,n}\,|\lambda_n\rangle\,. \tag{4.51}$$

Since each $X_{i,i+1}$ is a unitary operator acting on $\mathcal{H}_{\bar{A}}$, it is clear that $|\psi_i\rangle$ is a purification of $|\lambda_i\rangle$ for all i. The prefactors involving $S(\lambda_i,\lambda_i)$ ensure these states are normalised. Furthermore, we have

$$\langle\psi_{i+1}|\psi_i\rangle = e^{\frac{1}{2}S(\lambda_i,\lambda_i)}e^{\frac{1}{2}S(\lambda_{i+1},\lambda_{i+1})}\left\langle\lambda_{i+1}|X_{i,i+1}|\lambda_i\right\rangle \tag{4.52}$$

$$= \exp\left(\frac{1}{2}S(\lambda_i,\lambda_i)+\frac{1}{2}S(\lambda_{i+1},\lambda_{i+1})-S[\phi_{i,i+1}]+ix_{i,i+1}\right). \tag{4.53}$$

where the second line follows from the same logic as in (4.45), and we are assuming that

$$\int_{B_\Upsilon}\theta[\phi_{i,i+1},\delta\phi]=0. \tag{4.54}$$

The condition (4.54) may be enforced by a suitable redefinition $\theta\to\theta+\mathrm{d}K$, as described previously. In order for the bulk Hilbert space basis $|\varphi\rangle$ to be the same throughout the proceeding argument, this redefinition must be done using a K that is the same for all i. This is possible, and in fact one may enforce the stronger condition

$$\int_{B_\Upsilon}\theta[\phi,\delta\phi]=0, \tag{4.55}$$

where here $\phi,\delta\phi$ can be any field configuration and variation that might be discontinuous at $\bar{\Sigma}$, but obey the equations of motion elsewhere. Indeed, if (4.55) is not true, then we may pick a K such that

$$\int_{B_\Upsilon}\theta[\phi,\delta\phi]=-\int_{\partial B_\Upsilon}K[\phi,\delta\phi]. \tag{4.56}$$

In the limit as B_Υ tightly encloses Υ, both sides only depend on the fields at Υ. Thus we may choose K such that it only has a local dependence on the fields. After redefining $\theta\to\theta+\mathrm{d}K$, one then gets a θ satisfying (4.55).

One notes that $|\langle\psi_{i+1}|\psi_i\rangle|$ matches with the fidelity of ρ_i,ρ_{i+1}. Thus, each consecutive pair $|\psi_i\rangle,|\psi_{i+1}\rangle$ of the above states is parallel. One therefore can obtain the Uhlmann phase by computing the quantity

$$e^{i\gamma}=\lim_{n\to\infty}\langle\psi_1|\psi_n\rangle\langle\psi_n|\psi_{n-1}\rangle\dots\langle\psi_2|\psi_1\rangle\,. \tag{4.57}$$

All but the first factor in this object may be computed using (4.53). It thus remains to compute

$$\langle\psi_1|\psi_n\rangle=e^{\frac{1}{2}S(\lambda_1,\lambda_1)}e^{\frac{1}{2}S(\lambda_n,\lambda_n)}\left\langle\lambda_1|X^\dagger|\lambda_n\right\rangle, \tag{4.58}$$

where

$$X^\dagger=X^\dagger_{1,2}X^\dagger_{2,3}\dots X^\dagger_{n-1,n}. \tag{4.59}$$

Recall from the previous section that one may carry out a change of basis for the bulk Hilbert space by modifying the action by a boundary term $S_B = \int_B D$. It will be convenient for us to choose a $D[\phi]$ obeying the following condition for all i:

$$(\delta_i + \tilde{\delta}_i) \int_{\bar{\Sigma}^-} D[\phi_i] = -\int_{\bar{\Sigma}^-} \theta[\phi_i, \delta_i\phi + \tilde{\delta}_i\phi]. \tag{4.60}$$

If we view $\int_{\bar{\Sigma}^-} D$ as a function on field space, it is clear that we can satisfy this condition, because all we need to do is ensure that the derivative of this function at ϕ_i in the direction $\delta_i\phi + \tilde{\delta}_i\phi$ is given by the right-hand side. As mentioned previously, all the fields $\phi_i, \delta_i\phi, \tilde{\delta}_i\phi$ are invariant under $\mathbb{Z}_2 + \mathcal{C}$ symmetry.[10] Therefore, the right-hand side of (4.60) is imaginary (since $\mathbb{Z}_2$ changes the orientation of $\bar{\Sigma}^-$). Thus, this condition is consistent with the requirement that S_B be imaginary. Also, it is clear that this condition can be satisfied by a regular function S_B even as $n \to \infty$, because the vector $\delta\phi + \tilde{\delta}\phi$ is never parallel to the curve of field configurations ϕ. It may not be possible to choose D such that (4.60) is obeyed for *all* possible curves of field configurations, but this will end up not being important for our final result.

It is in principle unnecessary to enforce (4.60), and not doing so should still lead to the same results obtained in this chapter. However, we are free to enforce it, and this will be useful in what follows. For notational convenience, we will absorb δD into the definition of θ. Having done so, (4.60) means that we can assume

$$\int_{\bar{\Sigma}^-} \theta[\phi_i, \delta_i\phi + \tilde{\delta}_i\phi] = 0. \tag{4.61}$$

This then implies a certain choice of basis $|\varphi\rangle$ for the bulk Hilbert space.

Note that

$$\begin{aligned}
\varphi^+_{i,i+1} &= \text{boundary data for } \phi_i + \tfrac{1}{2}(\delta_i\phi + \tilde{\delta}_i\phi) \text{ at } \bar{\Sigma}^+ \\
&= \text{boundary data for } \phi_i + \tfrac{1}{2}(\delta_i\phi - \tilde{\delta}_i\phi) \text{ at } \bar{\Sigma}^- \\
&= \text{boundary data for } \phi_i + \delta_i\phi \text{ at } \bar{\Sigma}^- \\
&= \text{boundary data for } \phi_{i+1} \text{ at } \bar{\Sigma}^- \\
&= \text{boundary data for } \phi_{i+1} + \tfrac{1}{2}(\delta_{i+1}\phi + \tilde{\delta}_{i+1}\phi) \text{ at } \bar{\Sigma}^- \\
&= \varphi^-_{i+1,i+2},
\end{aligned} \tag{4.62}$$

where the third and fifth lines follow from (4.61). Therefore the two states $|\varphi^+_{i,i+1}\rangle$ and $|\varphi^-_{i+1,i+2}\rangle$ are actually equivalent. Thus, using

$$X^\dagger_{i,i+1} |\varphi^-_{i+1,i+2}\rangle = X^\dagger_{i,i+1} |\varphi^+_{i,i+1}\rangle = e^{-ix_{i,i+1}} |\varphi^-_{i,i+1}\rangle\,, \tag{4.63}$$

[10] To be clear, here we are considering the action of $\mathbb{Z}_2$ in such a way that it does *not* swap $\bar{\Sigma}^-$ and $\bar{\Sigma}^+$ – it merely applies a time reflection to all the components of the fields at $\bar{\Sigma}^-$.

we have

$$X^\dagger \left|\varphi^-_{n,1}\right\rangle = \exp\!\left(-i\sum_{i=1}^{n-1} x_{i,i+1}\right)\left|\varphi^-_{1,2}\right\rangle = \exp\!\left(-i\sum_{i=1}^{n-1} x_{i,i+1}\right)\left|\varphi^+_{n,1}\right\rangle . \tag{4.64}$$

Here we are defining $\varphi^\pm_{n,1}$ as the boundary data at $\Sigma \cup \bar{\Sigma}^\pm$ respectively for the field configuration $\phi_{n,1} + \frac{1}{2}(\delta_n\phi + \tilde{\delta}_n\phi)$, which is the one relevant to the fidelity of ρ_n, ρ_1. Because the curve of density matrices is closed, (4.62) applies for $\varphi^-_{n,1}, \varphi^+_{n,1}$ also, if we treat i as an index mod n.

The operator $X^\dagger$ corresponds to the symplectomorphism $Z^{-1}_{1,2}Z^{-1}_{2,3}\dots Z^{-1}_{n-1,n}$. At leading order in the limit $n \to \infty$, the operator $Z^{-1}_{i,i+1}$ can be treated as carrying out the infinitesimal change $\phi \to \phi - \tilde{\delta}_i\phi|_{\bar{\Sigma}^-}$. Thus $Z^{-1}_{1,2}Z^{-1}_{2,3}\dots Z^{-1}_{n-1,n}$ approximately acts as (in an appropriately linearised sense)

$$\phi \to \phi - \sum_{i=1}^{n-1} \tilde{\delta}_i\phi|_{\bar{\Sigma}^-}. \tag{4.65}$$

But note that

$$-\sum_{i=1}^{n-1} \tilde{\delta}_i\phi \approx \tilde{\delta}_n\phi. \tag{4.66}$$

This can be understood by considering the change in boundary conditions at asymptotic infinity for each $\tilde{\delta}_i\phi$. The fact that the curve of boundary conditions is closed implies that

$$\sum_{i=1}^{n} \delta_i\lambda \approx 0, \tag{4.67}$$

from which one obtains (4.66). Thus, at leading order in the limit $n \to \infty$, the operator $X^\dagger$ corresponds to a symplectomorphism Z^{-1} carrying out the change $\phi \to \phi + \tilde{\delta}_n\phi|_{\bar{\Sigma}}$. This maps $\phi_{n,1}|_{\bar{\Sigma}^-}$ to $\phi_{n,1}|_{\bar{\Sigma}^+}$.

Consider now the correlator

$$\langle\lambda_1|X^\dagger|\lambda_n\rangle = \int \mathrm{D}\phi_+\,\mathrm{D}\phi_-\, e^{-S^+[\phi_+]}X^\dagger e^{-S^-[\phi_-]}. \tag{4.68}$$

We sum over fields obeying the boundary conditions $\lambda_n, \lambda_1^{\mathrm{T}*}$ at $\mathcal{M}^-, \mathcal{M}^+$. By the same logic as in the previous section, we can use a stationary phase approximation to compute this integral. It is dominated by the field configuration which picks up an action of Z^{-1} when crossing B, and which obey the equations of motion elsewhere. This field configuration is $\phi_{n,1}$. Thus we have

$$\langle\lambda_1|X^\dagger|\lambda_n\rangle = e^{-S[\phi_{n,1}]}\left\langle\varphi^+_{n,1}\middle|X^\dagger\middle|\varphi^-_{n,1}\right\rangle . \tag{4.69}$$

By (4.64), the latter factor on the right-hand side is $\exp\!\left(-i\sum_{i=1}^{n-1} x_{i,i+1}\right)$.

Using this, we can compute the final scattering amplitude in the Uhlmann phase. In fact, with the choices we have made, it conveniently takes a form similar to all the other factors. It is

$$\langle\psi_1|\psi_n\rangle = \exp\!\left(\frac{1}{2}S(\lambda_1,\lambda_1) + \frac{1}{2}S(\lambda_n,\lambda_n) - S[\phi_{n,1}] - i\sum_{i=1}^{n-1} x_{i,i+1}\right). \tag{4.70}$$

In total, the Uhlmann phase may be written

$$e^{i\gamma} = \lim_{n\to\infty} \prod_{i=1}^{n} \exp\left(\frac{1}{2}S(\lambda_i,\lambda_i) + \frac{1}{2}S(\lambda_{i+1},\lambda_{i+1}) - S[\phi_{i,i+1}]\right), \tag{4.71}$$

where all the terms involving the numbers $x_{i,i+1}$ exactly cancel.

Let us compute the contribution of each term. Using

$$S(\lambda_i,\lambda_i) = S[\phi_i], \tag{4.72}$$

$$S(\lambda_{i+1},\lambda_{i+1}) = S[\phi_i] + \delta_i S[\phi_i] + \ldots, \tag{4.73}$$

$$S[\phi_{i,i+1}] = S[\phi_i] + \frac{1}{2}(\delta_i + \tilde{\delta}_i)S[\phi_i] + \ldots, \tag{4.74}$$

one finds

$$\frac{1}{2}S(\lambda_i,\lambda_i) + \frac{1}{2}S(\lambda_{i+1},\lambda_{i+1}) - S[\phi_{i,i+1}] = -\frac{1}{2}\tilde{\delta}_i S[\phi_i]. \tag{4.75}$$

In terms of boundary integrals, one has

$$-\frac{1}{2}\tilde{\delta}_i S[\phi_i] = -\frac{1}{2}\int_{\mathcal{M}^-\cup\mathcal{M}^+\cup\bar{\Sigma}^-\cup\bar{\Sigma}^+} \theta[\phi_i,\tilde{\delta}_i\phi] = -\int_{\mathcal{M}^-\cup\bar{\Sigma}^-} \theta[\phi_i,\tilde{\delta}_i\phi]. \tag{4.76}$$

Note that, at leading order in the variations, using (4.54) we may write

$$\int_{B_\Upsilon} \theta[\phi_i,\tilde{\delta}_i\phi] = \int_{B_\Upsilon} \theta[\phi_{i,i+1},\tilde{\delta}_i\phi] = 0, \tag{4.77}$$

which is why any terms at B_Υ in (4.76) can be neglected. The second equality in (4.76) comes from considering the action of $\mathbb{Z}_2 + \mathcal{C}$, where here $\mathbb{Z}_2$ refers to reflecting everything across $\Sigma\cup\bar{\Sigma}$. Because $\tilde{\delta}\phi$ picks up a minus sign from this transformation, and additionally the orientations of the integrals are flipped, we have

$$\int_{\mathcal{M}^+\cup\bar{\Sigma}^+} \theta[\phi_i,\tilde{\delta}_i\phi] = \int_{\mathcal{M}^-\cup\bar{\Sigma}^-} \theta[\phi_i,\tilde{\delta}_i\phi], \tag{4.78}$$

from which (4.76) follows.

Note that the holographic dictionary implies

$$\int_{\mathcal{M}^-} \theta[\phi,\delta\phi] = \int_{\mathcal{M}^-} \mathrm{d}\tau\,\mathrm{d}^{d-1}x\,\delta\lambda(\tau,x)\cdot\mathcal{O}(\tau,x). \tag{4.79}$$

Since the change in boundary conditions $\delta\lambda$ for $\delta_i\phi$ and $\tilde{\delta}_i\phi$ at $\mathcal{M}^-$ differ by a minus sign, we can write

$$-\int_{\mathcal{M}^-} \theta[\phi_i,\tilde{\delta}_i\phi] = \int_{\mathcal{M}^-} \theta[\phi_i,\delta_i\phi]. \tag{4.80}$$

Also, (4.60) implies that

$$-\int_{\bar{\Sigma}^-} \theta[\phi_i,\tilde{\delta}_i\phi] = \int_{\bar{\Sigma}^-} \theta[\phi_i,\delta_i\phi]. \tag{4.81}$$

Note that we may write $\bar{\Sigma}$ instead of $\bar{\Sigma}^-$ for the range of integration on the right-hand side, because the integrand is single-valued there. Therefore,

$$-\frac{1}{2}\tilde{\delta}_i S[\phi_i] = \int_{\mathcal{M}^- \cup \bar{\Sigma}} \theta[\phi_i, \delta_i \phi]. \tag{4.82}$$

Using (4.82) in (4.71), one finds

$$\gamma = -i \lim_{n\to\infty} \sum_{i=1}^{n} \int_{\mathcal{M} \cup \bar{\Sigma}} \theta[\phi_i, \delta_i \phi]. \tag{4.83}$$

The limit $n \to \infty$ may be carried out by replacing the sum by an integral. To be precise, we can write $\gamma = \oint_C a$, where

$$a = -i \int_{\mathcal{M}^- \cup \bar{\Sigma}} \theta \tag{4.84}$$

is a 1-form on the space of sources. This can be seen from the fact that it depends on the bulk field configuration and linearly on the variation of the bulk field configuration, so clearly it is a 1-form on the space of bulk field configurations. By pulling back this 1-form through the holographic map from boundary sources to bulk field configurations, we obtain the desired 1-form on the space of sources. $\gamma = \oint_C a$ is the Uhlmann phase of the curve C.

Note that we are free to redefine $a \to a + \delta\Lambda$, where Λ is any function on field space, and δ denotes an exterior derivative on field space. This is allowed since, by Stokes' theorem on field space, $\oint_C \delta\Lambda = 0$ for any Λ, and so the Uhlmann phase γ is unchanged. We will use this redefinition to put a in a slightly more natural form. Let

$$\Lambda[\phi] = i \int_{\mathcal{N}^-} L[\phi]. \tag{4.85}$$

We have

$$\delta\Lambda = i \int_{\mathcal{N}^-} \delta L = i \int_{\mathcal{N}^-} \mathrm{d}\theta = i \int_{\mathcal{M}^- \cup \bar{\Sigma} \cup \Sigma} \theta. \tag{4.86}$$

Thus, redefining $a \to a + \delta\Lambda$ yields

$$a = i \int_{\Sigma} \theta. \tag{4.87}$$

Before finishing this section, recall that one may add a boundary term $S_B = \int_B D$ to the action. The effect of such an addition is to change our final expression for a by

$$a \to a + i\delta\Big(\int_{\Sigma} D\Big). \tag{4.88}$$

But since this change is of the form $a \to a + \delta\Lambda$, it has no effect on the value of the Uhlmann phase. We previously mentioned that it might not be possible to satsify (4.60) by choosing the same S_B for all curves of states. It should be clear now that this is of no consequence, because different choices of S_B do not affect γ.

4.4 Symplectic form of the entanglement wedge

In the last section, we obtained the Uhlmann phase along a curve of reduced density matrices in a subregion A corresponding to boundary sources λ. We found that it was given by the integral of the 1-form a around that curve, where a is given in (4.87). In this section, we will treat a as a connection on the space of sources, for which the Uhlmann phase is the holonomy. From this point of view, $a \to a + \delta\Lambda$ is just a gauge transformation.

Let us compute the curvature of this connection. Using δ to again denote an exterior derivative on the space of sources, the curvature is given by

$$\Omega = \delta a = i\delta\left(\int_\Sigma \theta\right). \tag{4.89}$$

Since the δ is outside of the integral, we technically have to worry about field variations under which the location of the range of integration changes. In particular, the range of integration is determined dynamically by the fields, since Υ is the HRT surface. However, because theories of gravity are diffeomorphism invariant, we can always choose a gauge in which the range of integration is fixed.[11] We will do this for now for simplicity.[12] One has

$$\Omega = i\int_\Sigma \delta\theta. \tag{4.90}$$

The components of this 2-form with respect to two particular field variations $\delta_1\phi, \delta_2\phi$ is given by

$$i\int_\Sigma \omega[\phi, \delta_1\phi, \delta_2\phi], \tag{4.91}$$

where ω is defined in Section 1.1. Therefore, the curvature of the Uhlmann phase is equal to the integral of $i\omega$ over Σ.

[11] To add a bit more detail: diffeomorphisms can be considered from two points of view. In the active viewpoint, we move the fields ϕ. In the passive viewpoint, we deform the surface Σ itself. Diffeomorphism invariance is the statement that for each active change, there is a corresponding passive change and vice versa. If we do both the active and passive changes at the same time, i.e. if we move both the fields and the surface in the same way, all physical observations remain the same. So when diffeomorphism invariance is a feature of the theory, we can always apply a passive transformation to move the surface Σ to wherever we like, so long as we also do the corresponding active transformation. Here, we are simply doing this in such a way that the surface Σ always stays in the same place.

[12] If one wanted to consider the situation without this gauge-fixing, one would have to introduce degrees of freedom which track the location of Σ. This would lead to a formalism reminiscent of the 'extended phase space' of [62, 63, 136]. However, an important difference is as follows. In that paper, the fields ϕ and the location of Σ were more or less taken to be independent. In our setup, this is very much not the case, because $\partial\Sigma$ is dynamically determined by the fields.

It remains to consider the Lorentzian continuation of this result. Upon Wick rotation of the fields to Lorentzian signature, we have $i\int_\Sigma \omega \to \int_\Sigma \omega$, because there are an odd number of time derivatives in ω. Thus, the curvature of the Uhlmann phase is given by

$$\Omega = \int_\Sigma \omega_{\text{Lor}}, \tag{4.92}$$

where ω_{Lor} denotes ω evaluated on the Lorentzian fields. According to the covariant phase space formalism, Ω is exactly the symplectic form of the domain of dependence of Σ. Since Σ is bounded by A and the HRT surface Υ, the domain of dependence of Σ is the entanglement wedge of A. Thus, we come to the main result of the chapter:

> The curvature of the Uhlmann phase is holographically dual to the symplectic form of the entanglement wedge.

4.4.1 Subregion deformations and edge modes

Until now, we have considered the Uhlmann phase for a curve of density matrices in a fixed boundary subregion, but with varying sources. In this section, we will consider the case where we fix the sources, and vary the subregion.

Let $A(t)$, $0 \leq t \leq 1$ be a closed smooth curve in the space of boundary subregions. We assume for simplicity that this curve takes the form $A(t) = G_t(A)$, where G_t is a curve of boundary diffeomorphisms, and A is a fixed subregion. For fixed sources λ, we have a density matrix for each value of t given by reduction of the pure state $|\lambda\rangle$ to $A(t)$:

$$\rho(t) = \operatorname{tr}_{\overline{A(t)}} |\lambda\rangle \langle\lambda| \,. \tag{4.93}$$

We wish to compute the Uhlmann phase along this curve, but there is an obstruction to this, in that the density matrices for different values of t act on different Hilbert spaces $\mathcal{H}_{A(t)}$. In order to proceed, one must find appropriate maps from each of these Hilbert spaces to a common one, and it is not immediately obvious which maps these should be.

It has been argued by various authors [18, 51, 65, 66, 113] that infinitesimal deformations of the boundary subregion may alternatively be thought of as being sourced by appropriate insertions of the stress-tensor.[13] In particular, if we want to understand the change induced by a

[13] Note that for this interpretation to be correct, the generic result of these insertions must be a non-unitary change of the state ρ, since an arbitrary deformation can move information across the boundary. It seems OK to throw away information that leaves the boundary, but where does the information that enters the boundary come from – is this a

deformation generated by the vector field ζ, one may insert

$$\mathcal{L}_\zeta g_{ab} T^{ab} \tag{4.94}$$

in the boundary state, where g_{ab} is the boundary metric, and T^{ab} is the boundary stress-tensor. One may view this as a change $\mathcal{L}_\zeta g_{ab}$ in the source of the operator T^{ab}. We will write $\lambda \to \lambda + \delta_\zeta \lambda$ to represent this change.

With this interpretation one can construct density matrices acting on the same Hilbert space. In particular one obtains infinitesimally close density matrices

$$\rho = \mathrm{tr}_{\bar{A}} \left|\lambda\right\rangle \left\langle\lambda\right| , \tag{4.95}$$

$$\rho' = \mathrm{tr}_{\bar{A}} \left|\lambda + \delta_\zeta \lambda\right\rangle \left\langle\lambda + \delta_\zeta \lambda\right| , \tag{4.96}$$

which both act on $\mathcal{H}_A$. By integrating this construction along the whole curve of subregions $A(t)$, we get a curve of density matrices

$$\rho(t) = \mathrm{tr}_{\bar{A}} \left|\lambda(t)\right\rangle \left\langle\lambda(t)\right| , \tag{4.97}$$

all acting on $\mathcal{H}_A$. Here $\lambda(t)$ obeys $\lambda(0) = \lambda$ and

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = \delta_\zeta \lambda, \tag{4.98}$$

where ζ is the infinitesimal vector field that generates evolution along the curve of diffeomorphisms G_t.

With this prescription, we can now compute the Uhlmann phase, and it should be clear that this is just a special case of what we have been considering previously, because the subregion is fixed to A, and we are varying the sources.

Diffeomorphism invariance makes it easy to compute the bulk fields corresponding to these sources. Let ϕ be the bulk field configuration matching the boundary conditions set by λ, and let H_t be a curve of diffeomorphisms acting on the bulk with the property that H_t restricted to the boundary is equal to G_t. Then $\phi(t) = H_t^* \phi$ is the bulk field configuration matching the boundary

problem? Our tentative belief is no. The form of the insertion (4.94) depends on the fields at the boundary, and so will infinitesimally be enough to supply the right information. A finite deformation will result from an integration over many such insertions, and this integrated insertion depends on the fields in a finite region outside of the subregion. So the deformation depends on the state outside of the subregion in question, and this is where the extra information that generates the non-unitary evolution comes from.

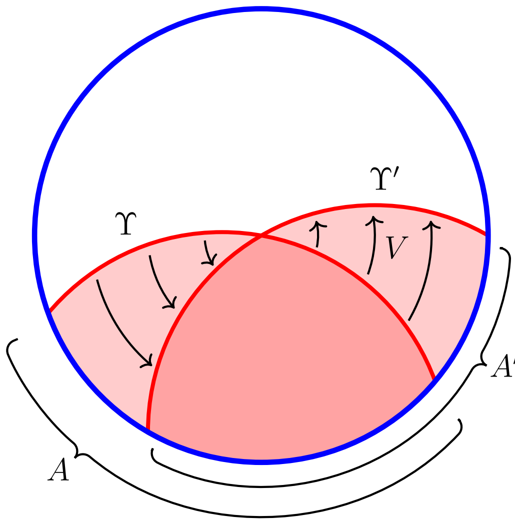


Figure 4.11: A deformation of the boundary subregion $A \to A'$ leads to a corresponding deformation of the HRT surface $\Upsilon \to \Upsilon'$, generated by a bulk vector field V.

conditions set by $\lambda(t)$. The field variation along this curve is given by $\delta\phi = \mathcal{L}_V\phi$, where V is the bulk vector field corresponding to infinitesimal evolution along H_t.

Each boundary subregion $A(t)$ has an associated HRT surface $\Upsilon(t)$ in the bulk. Recall that we have fixed the gauge such that Σ must have its boundary at the HRT surface. In order for H_t to be consistent with this gauge choice, it must be the case that $H_t(\Upsilon) = \Upsilon(t)$. Thus, the vector field V at Υ generates the deformation of the HRT surface $\Upsilon(t)$. This is depicted in Figure 4.11.

Using the results of Section 1.1, we can write down the components of the symplectic form with respect to this deformation,

$$\Omega[\phi, \mathcal{L}_V\phi, \delta\phi] = -\int_{\partial\Sigma} \Big(\delta Q_V[\phi] - Q_{\delta V}[\phi] + \iota_V\theta[\phi, \delta\phi]\Big), \tag{4.99}$$

where Q_V was defined in (1.27). Thus, the components of the curvature of the Uhlmann phase with respect to a subregion deformation reduce to a boundary integral at $\partial\Sigma$.

This feature of the covariant phase space formalism is not unique to our situation, and has been observed many times before [21, 22, 39, 48, 127, 153–155]. However, there has previously not been much reason to restrict the form which V can take.[14] In our case, V is much more constrained – it must be a vector field representing an infinitesimal deformation of one HRT

[14] Sometimes, boundary conditions have been imposed at $\partial\Sigma$. Then V must preserve these boundary conditions. Usually, however, the boundary conditions are somewhat arbitrary, and no a priori justification for them is given.

surface to another nearby HRT surface.[15] Thus, we get a classical degree of freedom, living at $\partial\Sigma$, for each such deformation. Such degrees of freedom are referred to as edge modes.

For some deformations, it may be the case that we can write

$$\Omega[\phi, \mathcal{L}_V\phi, \delta\phi] = -\delta H_V, \tag{4.100}$$

where $H_V = H_V[\phi]$ is some function on field space. Then the deformation is integrable, and H_V is the Hamiltonian generating the deformation. It is an interesting question to ask which deformations are integrable. For now let us simply comment that this question is intimately associated with the conformal symmetry of the boundary theory. Indeed, consider the case where the subregion A contains the entire boundary. Then the HRT surface is empty, and $\Omega[\phi, \mathcal{L}_V\phi, \delta\phi]$ is expressible entirely in terms of the fields on the boundary, and in this case the question reduces to asking when the curve of states $|G_t^*\lambda\rangle$ can be written as $e^{iHt}|\lambda\rangle$, for some boundary operator H. Of course, the answer is that this is the case when G_t is a conformal transformation. Then H is simply the generator of that conformal transformation. When A is a proper subregion of the boundary, the situation becomes more complicated due to the fact that $\Omega[\phi, \mathcal{L}_V, \delta\phi]$ contains terms at the HRT surface. But clearly the conformal transformations are a good place to start.

Because $\Omega[\phi, \mathcal{L}_V\phi, \delta\phi]$ contains contributions at the HRT surface, it is in principle possible to use the Uhlmann phase arising from deformations of the boundary subregion A to measure the fields near the HRT surface, including the Riemann curvature. Similar conclusions were drawn in [51]. However, in that paper the authors discussed a different type of phase, which they called the modular Berry phase. It would be interesting to understand the relationship between the Uhlmann phase and the modular Berry phase in the holographic context.

4.4.2 Resolution of the boundary ambiguity

Recall from the Introduction that there is an ambiguity in the definition of θ. In particular one is allowed to modify θ by any exact form. This changes the integral $\int_\Sigma \theta$ by a boundary term at $\partial\Sigma$. It is clear that one must fix this ambiguity if one is to understand the degrees of freedom near the boundary.

In Section 4.3.3, we used this freedom to enforce the condition

$$\int_{B_\Upsilon} \theta[\phi_{i,i+1}, \delta\phi] = 0. \tag{4.101}$$

[15] In the language of the previous footnote, we now have the non-arbitrary boundary condition that Υ must be an HRT surface.

We will now show that this condition fixes the boundary ambiguity at Υ.

Suppose (4.101) holds, and we attempt to modify $\theta \to \theta + \mathrm{d}K$. Then we would have

$$0 = \int_{B_\Upsilon} \theta[\phi_{i,i+1}, \delta\phi] \to \int_{B_\Upsilon} \theta[\phi_{i,i+1}, \delta\phi] + \int_{\partial B_\Upsilon} K[\phi_{i,i+1}, \delta\phi]. \tag{4.102}$$

Clearly (4.101) only continues to hold if the integral over ∂B_Υ vanishes. ∂B_Υ has two components $\Upsilon^\pm = \partial\bar{\Sigma}^\pm \cap B_\Upsilon$. We thus require

$$0 = \int_{\Upsilon^+} K[\phi_{i,i+1}, \delta\phi] - \int_{\Upsilon^-} K[\phi_{i,i+1}, \delta\phi]. \tag{4.103}$$

In order for our expression of the Uhlmann curvature to be valid in all situations, we want this to hold for *any* possible $\phi_{i,i+1}$ and $\delta\phi$ which obey the equations of motion everywhere but at $\bar{\Sigma}$. The contributions of these fields at Υ^- and Υ^+ are essentially independent of one another, and freely specifiable in the above. Thus, we need

$$0 = \int_{\Upsilon^\pm} K[\phi, \delta\phi] \tag{4.104}$$

for any $\phi, \delta\phi$. In the limit as B_Υ tightly encloses Υ, we can replace $\Upsilon^\pm \to \Upsilon$, and so obtain

$$0 = \int_\Upsilon K[\phi, \delta\phi]. \tag{4.105}$$

But note that $\theta \to \theta + \mathrm{d}K$ for such a K implies

$$\int_\Sigma \theta \to \int_\Sigma \theta + \int_\Upsilon K[\phi, \delta\phi] = \int_\Sigma \theta. \tag{4.106}$$

Here we are assuming that the ambiguity is fixed at asymptotic infinity in some other way, so there is no contribution from K there. Thus, any change $\theta \to \theta + \mathrm{d}K$ that respects (4.101) must lead to no change in $\int_\Sigma \theta$. So, this condition does indeed fix the ambiguity at the HRT surface.

The condition (4.101) is formulated as a Euclidean expression. However, for practical purposes it would be more convenient to be able to state it in terms of the fields in the Wick rotated Lorentzian bulk spacetime. Also, it would be useful to see if (4.101) could be understood in a simple and convenient way for some basic examples, such as pure Einstein gravity. We leave these and other questions for future work.

4.5 Discussion

In this chapter we have argued that the holographic dual of the symplectic form in an entanglement wedge is the curvature of the Uhlmann phase for states reduced to the corresponding boundary subregion. Let us briefly speculate on some consequences and possible future applications of this result.

First, our result gives a specific operational context to the concept of classically emergent physics in a subregion that was previously somewhat absent: classical bulk subregion physics emerges in measurements of the Uhlmann phase. It is important to point out that the Uhlmann phase is a genuine observable, as has been argued in principle [2], and has recently has been confirmed in practice [151]. It would be useful to figure out more of the details of this context.

Second, we would like to more fully understand the resolution of the boundary ambiguity for the symplectic form given in this chapter, and its implications for edge modes. For example, the edge modes have been used to try to understand black hole entropy [71, 72], and it would be worthwhile to see if the methods used in those papers are consistent with our results.

Third, starting from the quantum mechanical description of a complete holographic system, our construction resulted in a classical phase space for the degrees of freedom in the entanglement wedge. It is natural to attempt to run this backwards, i.e. to quantise this phase space. One would then obtain an 'effective' Hilbert space for the entanglement wedge. In the original system, all the states in the entanglement wedge had to be mixed, because of entanglement in the CFT. However, the effective entanglement wedge Hilbert space is clearly made up of *pure* states. Hence, by studying such a quantisation, one should be able to learn about what it means to have a pure state in a gravitational subregion. This would be of particular interest in the case of a black hole spacetime, where the entanglement wedge is chosen to coincide with the black hole exterior. The pure states in the effective Hilbert space might then reasonably be called black hole microstates.

Fourth, the calculation presented in this chapter applies at leading order for large N. It would be interesting to try to understand the subleading corrections, where the condition on the HRT surface is supposed to be changed from extremising the area, to extremising the generalised entropy [68].

Fifth, recent progress on the information paradox has been made by considering the contributions of so-called 'islands' in replica path integrals [9, 10, 126]. It seems likely that these islands will contribute to the symplectic structure described here. After the Page time, this would imply that there are contributions to black hole soft hair from the boundaries of islands. It may be possible to use this idea to understand what happens to soft hair during evaporation.

Sixth, in [27] the holographic Berry curvature was used to investigate the complexity of holographic states. It may be possible to use our results to extend that analysis to holographic subregion complexity, which has previously been explored in [3, 4, 6, 29, 45–47].

Seventh, Uhlmann phases have been used to classify phases of condensed matter systems [152]. It would be interesting to see if our expression for the Uhlmann phase could be used in a similar way, in the cases where the systems have holographic duals.

Finally, we should note that throughout this chapter we have been working in terms of density matrices acting on well-defined Hilbert spaces for subregions. However, technically speaking, in a QFT such Hilbert spaces do not exist, and a more rigorous treatment would involve type III von Neumann algebras. It is only after an appropriate UV regularisation that one can hope to involve Hilbert spaces. This is commonly done – one assumes that such a regularisation has taken place. It would be useful to see how the regularisation procedure interacts with the results we have obtained, and in particular to see how the symplectic structure changes under holographic renormalisation. Also, it may be worthwhile to see if it is possible to define Uhlmann holonomy purely in terms of von Neumann algebras, forgoing density matrices completely.

Chapter 5

Uhlmann Phase and Emergent Holography

5.1 Introduction

In the previous chapter, we showed how one may compute the Uhlmann phase of a holographic system using semiclassical approximations in a gravitational path integral. In this chapter, we will demonstrate that one may obtain a somewhat dual result if one approaches these ideas from a slightly different direction. In particular, we will derive a path integral formula for the Uhlmann phase along a curve of reduced density matrices in a *generic* system. Suprisingly, when the system is highly entangled, the path integral takes on many holographic characteristics, despite the fact that (in this chapter) we make no assumptions about any pre-existing holographic mechanism. This suggests that there are deep links between Uhlmann holonomy, entanglement, and holography.

We obtain the path integral formula itself in Section 5.2. The first half of that Section, up to (5.34), reviews material already known to Uhlmann [145–147], but the rest of it is new. Then, in Section 5.3, we describe when exactly the path integral has a classical limit. In Section 5.4 we explain why, for sufficiently entangled states, the path integral should be viewed as a holographic one. We end with some brief discussion on the broader role of Uhlmann holonomy in holography.

Let us provide a slightly more detailed roadmap for the calculations that follow. First, we will write the Uhlmann phase for a given path of density matrices $\rho(t)$ in terms of the expectation value of a path ordered exponential (5.34) along that path. The 1-form A that is integrated is defined in (5.33), and is known as the Uhlmann connection. To facilitate the holographic interpretation, it is useful for us to write the Uhlmann connection in terms of 'modular flow', and we do this next, obtaining (5.54). Then, we write the path ordered exponential (5.34) as a path integral in the usual way, i.e. by inserting many resolutions of the identity $I = \int \mathrm{d}x \, |x\rangle \langle x|$ and taking a continuum limit. Here, we assume the states $|x\rangle$ are coherent states. Using the modular flow representation of the Uhlmann connection, we are able to write the action for this path integral in terms of a natural set of states $|x(t,\alpha)\rangle$, which are related to each other by modular flow. The action we obtain is

$$\mathbf{S}[x] = i\hbar \int_0^1 \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{sech}(\pi\alpha) \, \langle x(t,\alpha)| \frac{\partial}{\partial t} |x(t,\alpha)\rangle \, . \tag{5.1}$$

Comparing this to the usual action for the transition amplitude of coherent states, and taking the classical limit, we argue that the coordinate α spans a new emergent dimension when the density matrices $\rho(t)$ are sufficiently mixed, which means $-\log \rho(t) = \mathcal{O}(1/\hbar)$. We make this precise by recognizing (5.1) as a Hamiltonian action, and identifying the symplectic form (5.96).

The behaviour we describe appears to be non-trivial and interesting, but we should note that the holographic interpretation we are putting forward is at this point somewhat speculative. We will explain why we believe it is an appropriate one. However, not all of the details have been worked out, and we will try to also highlight these missing pieces.

5.2 The path integral formula

The starting point is a quantum system with Hilbert space $\mathcal{H}$. Suppose the state of the system is described by an invertible density matrix ρ acting on $\mathcal{H}$. It is useful to view ρ as arising from the reduction of some state $|\Psi\rangle$ in an extended Hilbert space $\mathcal{H} \otimes \mathcal{H}^*$ to the first factor $\mathcal{H}$. There are many possible choices of $|\Psi\rangle$, known as purifications of ρ.

Let $|\Psi\rangle$ be some purification of ρ. By dualising on the $\mathcal{H}^*$ factor, we may view $|\Psi\rangle$ instead as a linear map $W : \mathcal{H} \to \mathcal{H}$. The condition that $|\Psi\rangle$ purifies ρ may then be conveniently written as

$$\rho = WW^\dagger. \tag{5.2}$$

A polar decomposition of W allows us to write

$$W = \sqrt{\rho}U, \tag{5.3}$$

where U is some unitary operator. Actually, because we are assuming ρ is invertible, U is uniquely determined by W. Thus, the choice of U is a one-to-one parametrisation of the purifications of ρ.

5.2.1 Uhlmann phase as an expectation value

Recall that in computing an Uhlmann phase one is first given a closed curve $\rho(t)$, $0 \leq t < 1$, of mixed states acting on $\mathcal{H}$. One then is tasked with finding states $|\Psi_i\rangle \in \mathcal{H} \otimes \mathcal{H}^*$ at $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ for which $|\Psi_i\rangle$ purifies $\rho(t_i)$ for all i, and which maximise the transition probability

$$|\langle \Psi_{i+1} | \Psi_i \rangle|^2. \tag{5.4}$$

These conditions fully determine the states $|\Psi_i\rangle$, up to phase shifts.

Let us view each $|\Psi_i\rangle$ as a linear map $W_i : \mathcal{H} \to \mathcal{H}$, with polar decomposition

$$W_i = \sqrt{\rho_i} U_i, \tag{5.5}$$

where $\rho_i = \rho(t_i)$. The phase ambiguity of $|\Psi_i\rangle$ is now reflected in the possibility of a phase shift $U_i \to U_i e^{i f_i}$. In terms of the U_i, we have

$$|\langle \Psi_{i+1} | \Psi_i \rangle|^2 = \left| \mathrm{tr}\left(U_{i+1}^\dagger \sqrt{\rho_{i+1}} \sqrt{\rho_i} U_i \right) \right|^2. \tag{5.6}$$

One then takes a continuum limit in which $n \to \infty$, the sequence t_i densely covers the range of the parameter t, and the purifications $|\Psi_i\rangle$ converge to a curve $|\Psi(t)\rangle$ obeying $|\Psi(t_i)\rangle = |\Psi_i\rangle$. In terms of the maps W_i, we get a curve $W(t)$ of such maps with polar decompositions

$$W(t) = \sqrt{\rho(t)} U(t), \tag{5.7}$$

for some curve of unitary operators $U(t)$.

The operators U_i and so the curve $U(t)$ are only defined up to phase shifts. However, any phase ambiguities cancel when we compute the Uhlmann phase γ, which is defined via

$$e^{i\gamma} = \lim_{n \to \infty} \langle \Psi(t_0) | \Psi(t_n) \rangle \ldots \langle \Psi(t_2) | \Psi(t_1) \rangle \langle \Psi(t_1) | \Psi(t_0) \rangle . \tag{5.8}$$

So the phase shifts can be viewed as a kind of gauge transformation, and the Uhlmann phase is gauge invariant.

It will be convenient for us to make a particular gauge choice. The choice we make is for

$$\langle \Psi(t_{i+1}) | \Psi(t_i) \rangle \tag{5.9}$$

to be real and positive for all $0 \leq i < n$, which can clearly always be made to be true by appropriate phase shifts. Any remaining phase shifts which preserve this condition must act in the same way on all $|\Psi_i\rangle$, i.e. must be of the form $|\Psi_i\rangle \to e^{if}|\Psi_i\rangle$ for a constant f. Similarly, for the operators U_i, any phase shift must be of the form $U_i \to U_i e^{if}$.

The Uhlmann phase is invariant under reparametrisations with respect to t, so without loss of generality we can set $t_i = i/n$. In the $n \to \infty$ continuum limit, we can then write

$$|\Psi(t_{i+1})\rangle = |\Psi(t_i)\rangle + \frac{1}{n}|\dot{\Psi}(t_i)\rangle + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{5.10}$$

where a dot denotes differentiation with respect to s. Then we have

$$\langle\Psi(t_{i+1})|\Psi(t_i)\rangle = 1 + \frac{1}{n}\langle\dot{\Psi}(t_i)|\Psi(t_i)\rangle + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{5.11}$$

However, note that the second term on the right hand side is imaginary. Thus, with the gauge choice made above, this term must vanish. Hence we can write

$$\langle\Psi(t_{i+1})|\Psi(t_i)\rangle = 1 + \mathcal{O}\left(\frac{1}{n^2}\right) = \exp\left(\mathcal{O}\left(\frac{1}{n^2}\right)\right). \tag{5.12}$$

As $n \to \infty$ we therefore have

$$\begin{aligned}&\langle\Psi(t_n)|\Psi(t_{n-1})\rangle \ldots \langle\Psi(t_2)|\Psi(t_1)\rangle\,\langle\Psi(t_1)|\Psi(t_0)\rangle\\ &\qquad\sim \prod_{i=0}^{n-1}\exp\left(\mathcal{O}\left(\frac{1}{n^2}\right)\right) = \exp\left(\sum_{i=0}^{n-1}\mathcal{O}\left(\frac{1}{n^2}\right)\right) = \exp(\mathcal{O}(1/n)) \to 1.\end{aligned} \tag{5.13}$$

So, with this gauge choice, the definition of the Uhlmann phase reduces to

$$\begin{aligned}e^{i\gamma} &= \lim_{n\to\infty}\langle\Psi(t_0)|\Psi(t_n)\rangle = \langle\Psi(0)|\Psi(1)\rangle\\ &= \operatorname{tr}\left(U(0)^\dagger\sqrt{\rho(0)}\sqrt{\rho(1)}U(1)\right) = \operatorname{tr}\left(\rho(0)U(1)U(0)^\dagger\right).\end{aligned} \tag{5.14}$$

Thus, the Uhlmann phase may be written as the expectation value of the operator $U(1)U(0)^\dagger = \lim_{n\to\infty} U_n U_0^\dagger$ in the state $\rho(0)$. Reassuringly, this operator is invariant under the remaining allowed phase shifts $U_i \to U_i e^{if}$.

5.2.2 Uhlmann connection

In terms of the operators U_i, the gauge choice we have made is that

$$\operatorname{tr}\left(U_{i+1}^\dagger\sqrt{\rho_{i+1}}\sqrt{\rho_i}U_i\right) \tag{5.15}$$

should be real and positive. We also have to maximise

$$\left|\operatorname{tr}\left(U_{i+1}^\dagger\sqrt{\rho_{i+1}}\sqrt{\rho_i}U_i\right)\right|^2 \tag{5.16}$$

with respect to U_{i+1} and U_i. Clearly, these conditions will be satisfied if we choose U_{i+1} and U_i so that the eigenvalues of

$$U_i U_{i+1}^\dagger \sqrt{\rho_{i+1}}\sqrt{\rho_i} \tag{5.17}$$

all lie on the positive real axis of the complex plane.

To show that this is possible, it suffices to use another polar decomposition. Let us write

$$\sqrt{\rho_{i+1}}\sqrt{\rho_i} = V_i \sqrt{\sqrt{\rho_i}\rho_{i+1}\sqrt{\rho_i}}, \tag{5.18}$$

for a unitary operator V_i. Because $\sqrt{\sqrt{\rho_i}\rho_{i+1}\sqrt{\rho_i}}$ is invertible, V_i is uniquely determined. Then if we set $U_{i+1}U_i^\dagger = V_i$, we have

$$U_i U_{i+1}^\dagger \sqrt{\rho_{i+1}}\sqrt{\rho_i} = \sqrt{\sqrt{\rho_i}\rho_{i+1}\sqrt{\rho_i}}. \tag{5.19}$$

This is a positive operator, so we satisfy the conditions above.

In the continuum limit, V_i will be very close to the identity, so let us write $V_i = e^{B_i}$, where B_i is an anti-Hermitian operator which goes to 0 as $n \to \infty$. The fact that (5.17) is Hermitian when $U_{i+1}U_i^\dagger = V_i$ suffices to determine the leading order part of B_i. Indeed, using

$$\sqrt{\rho_{i+1}} = \sqrt{\rho_i} + \frac{1}{n}\dot{\sqrt{\rho(t_i)}} + \mathcal{O}\Big(\frac{1}{n^2}\Big), \tag{5.20}$$

we have

$$\begin{aligned}
0 &= U_i U_{i+1}^\dagger \sqrt{\rho_{i+1}}\sqrt{\rho_i} - (U_i U_{i+1}^\dagger \sqrt{\rho_{i+1}}\sqrt{\rho_i})^\dagger && (5.21)\\
&= V_i^\dagger \sqrt{\rho_{i+1}}\sqrt{\rho_i} - \sqrt{\rho_i}\sqrt{\rho_{i+1}}V_i && (5.22)\\
&= \frac{1}{n}\Big(\dot{\sqrt{\rho(t_i)}}\sqrt{\rho(t_i)} - \sqrt{\rho(t_i)}\dot{\sqrt{\rho(t_i)}}\Big) - B_i\rho(t_i) - \rho(t_i)B_i + \mathcal{O}\Big(\frac{1}{n^2}\Big) + \mathcal{O}\big(B_i^2\big). && (5.23)
\end{aligned}$$

At leading order this is solved by

$$B_i = \frac{1}{n}a(t_i) + \mathcal{O}\Big(\frac{1}{n^2}\Big), \tag{5.24}$$

if we can find an anti-Hermitian operator $a(t)$ solving

$$a\rho + \rho a = \dot{\sqrt{\rho}}\sqrt{\rho} - \sqrt{\rho}\dot{\sqrt{\rho}}. \tag{5.25}$$

In fact, we can find such an operator:

$$a = \int_0^\infty \mathrm{d}s\, e^{-s\rho}\Big(\dot{\sqrt{\rho}}\sqrt{\rho} - \sqrt{\rho}\dot{\sqrt{\rho}}\Big)e^{-s\rho}. \tag{5.26}$$

The integral converges because ρ is positive, and it may be confirmed that this a solves the required equation by direct substitution; the anticommutator with ρ converts to a total derivative in the integral.

We now have

$$U(1)U(0)^\dagger = \lim_{n\to\infty} U_n U_0^\dagger \tag{5.27}$$

$$= \lim_{n\to\infty} U_n U_{n-1}^\dagger, U_{n-1} U_{n-2}^\dagger \dots U_1 U_0^\dagger \tag{5.28}$$

$$= \lim_{n\to\infty} V_{n-1} V_{n-2} \dots V_0 \tag{5.29}$$

$$= \lim_{n\to\infty} e^{\frac{1}{n} a(t_{n-1}) + \mathcal{O}\left(\frac{1}{n^2}\right)} \dots e^{\frac{1}{n} a(t_0) + \mathcal{O}\left(\frac{1}{n^2}\right)} \tag{5.30}$$

$$= \mathrm{P}\exp\left(\int_0^1 a(t)\,\mathrm{d}t\right). \tag{5.31}$$

The expression in the last line is a path ordered exponential. The Uhlmann phase may now be written

$$e^{i\gamma} = \mathrm{tr}\left(\rho(0)\,\mathrm{P}\exp\left(\int_0^1 a(t)\,\mathrm{d}t\right)\right). \tag{5.32}$$

Let us define an anti-Hermitian operator valued 1-form A on the space of density matrices by

$$A = \int_0^\infty \mathrm{d}s\, e^{-s\rho}\left(\,\mathrm{d}(\sqrt{\rho})\,\sqrt{\rho} - \sqrt{\rho}\,\mathrm{d}(\sqrt{\rho})\,\right)e^{-s\rho}. \tag{5.33}$$

This 1-form is the 'Uhlmann connection'. It is clear that $a(t)\,\mathrm{d}t$ is the component of A along the curve $\rho(t)$. This allows us to write the Uhlmann phase in the more geometrically natural way

$$e^{i\gamma} = \mathrm{tr}\left(\rho(0)\,\mathrm{P}\exp\left(\int_C A\right)\right), \tag{5.34}$$

where C is the curve of density matrices.

5.2.3 Modular flow

It will actually be more convenient for us to write a in a different form, using modular flow. Given a density matrix $\rho : \mathcal{H} \to \mathcal{H}$, modular flow is a one-parameter automorphism of the algebra of operators acting on $\mathcal{H}$. It is defined by

$$O \mapsto \rho^{i\alpha} O \rho^{-i\alpha}, \tag{5.35}$$

where α is the parameter.

In our case, we have a 1-parameter family of density matrices labelled by t. We can do modular flow with any of these density matrices. So overall there are two parameters, t and α.

Let the spectral decomposition of the modular Hamiltonian $K = -\log\rho$ be given by[1]

$$K = \int_{-\infty}^\infty E\,\mathrm{d}\Pi_E\,. \tag{5.36}$$

[1] Technically, we could change the lower limit in this integral to 0, because $K > 0$.

Here $\mathrm{d}\Pi_E$ is a projection valued measure, defined such that

$$\Pi_{[E_1,E_2]} = \int_{E_1}^{E_2} \mathrm{d}\Pi_E \tag{5.37}$$

is the projector onto the space spanned by states with modular energy (i.e. K eigenvalue) in the range $[E_1, E_2]$. The identity and ρ may be written in terms of this measure as

$$I = \int_{-\infty}^{\infty} \mathrm{d}\Pi_E \tag{5.38}$$

$$\rho = \int_{-\infty}^{\infty} e^{-E}\, \mathrm{d}\Pi_E\,. \tag{5.39}$$

It is useful to note the explicit formula

$$\frac{\mathrm{d}\Pi_E}{\mathrm{d}E} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\alpha\, e^{i\alpha E}\rho^{i\alpha}, \tag{5.40}$$

which is just a Fourier transform. The presence of $\rho^{i\alpha}$ is what enables us to make the connection with modular flow.

Acting with (5.38) on the left and right of (5.26), we get

$$a = \int_{-\infty}^{\infty} \mathrm{d}\Pi_E \left(\dot{\sqrt{\rho}}\sqrt{\rho} - \sqrt{\rho}\dot{\sqrt{\rho}}\right) \int_{-\infty}^{\infty} \mathrm{d}\Pi_{\tilde{E}} \int_0^{\infty} \mathrm{d}s \exp\!\left(-se^{-E}\right) \exp\!\left(-se^{-\tilde{E}}\right) \tag{5.41}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}\Pi_E\, \dot{\sqrt{\rho}} \int_{-\infty}^{\infty} \mathrm{d}\Pi_{\tilde{E}}\, \frac{e^{-\frac{1}{2}\tilde{E}} - e^{-\frac{1}{2}E}}{e^{-E} + e^{-\tilde{E}}} \tag{5.42}$$

Note that for integer m we have

$$\mathrm{d}\Pi_E\, (\dot{\rho^m})\, \mathrm{d}\Pi_{\tilde{E}} = \sum_{j=0}^{m-1} \mathrm{d}\Pi_E\, \rho^m \dot{\rho} \rho^{m-1-j}\, \mathrm{d}\Pi_{\tilde{E}} \tag{5.43}$$

$$= \mathrm{d}\Pi_E\, \dot{\rho}\, \mathrm{d}\Pi_{\tilde{E}} \sum_{j=0}^{m-1} e^{-jE} e^{-(m-1-j)\tilde{E}} \tag{5.44}$$

$$= \mathrm{d}\Pi_E\, \dot{\rho}\, \mathrm{d}\Pi_{\tilde{E}}\, \frac{e^{-mE} - e^{-m\tilde{E}}}{e^{-E} - e^{-\tilde{E}}}. \tag{5.45}$$

By analytic continuation of m, we have

$$\mathrm{d}\Pi_E\, \dot{K}\, \mathrm{d}\Pi_{\tilde{E}} = -\frac{\mathrm{d}}{\mathrm{d}m}\, \mathrm{d}\Pi_E\, (\dot{\rho^m})\, \mathrm{d}\Pi_{\tilde{E}}\Big|_{m=0} \tag{5.46}$$

$$= -\,\mathrm{d}\Pi_E\, \dot{\rho}\, \mathrm{d}\Pi_{\tilde{E}}\, \frac{\mathrm{d}}{\mathrm{d}m} \frac{e^{-mE} - e^{-m\tilde{E}}}{e^{-E} - e^{-\tilde{E}}}\Bigg|_{m=0} \tag{5.47}$$

$$= \mathrm{d}\Pi_E\, \dot{\rho}\, \mathrm{d}\Pi_{\tilde{E}}\, \frac{E - \tilde{E}}{e^{-E} - e^{-\tilde{E}}}. \tag{5.48}$$

We can combine these to write

$$\mathrm{d}\Pi_E\, \dot{\sqrt{\rho}}\, \mathrm{d}\Pi_{\tilde{E}} = \mathrm{d}\Pi_E\, \dot{K}\, \mathrm{d}\Pi_{\tilde{E}}\, \frac{e^{-\frac{1}{2}E} - e^{-\frac{1}{2}\tilde{E}}}{E - \tilde{E}}. \tag{5.49}$$

Substituting this into the above, one finds

$$a = \int_{-\infty}^{\infty} \mathrm{d}\Pi_E \, \dot{K} \int_{-\infty}^{\infty} \mathrm{d}\Pi_{\tilde{E}} \, \frac{\operatorname{sech}\Big((E-\tilde{E})/2\Big) - 1}{E - \tilde{E}} \tag{5.50}$$

Let us now use (5.40) in this equation. We get

$$a = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}E \int_{-\infty}^{\infty} \mathrm{d}\tilde{E} \int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} \mathrm{d}\tilde{\alpha} \, e^{i\alpha E} e^{i\tilde{\alpha}\tilde{E}} \rho^{i\alpha} \dot{K} \rho^{i\tilde{\alpha}} \frac{\operatorname{sech}\Big((E-\tilde{E})/2\Big) - 1}{E - \tilde{E}}. \tag{5.51}$$

Things simplify at this point if we change variables from $E, \tilde{E}$ to

$$x = \frac{1}{2}(E + \tilde{E}), \qquad y = \frac{1}{2}(E - \tilde{E}), \tag{5.52}$$

so that

$$a = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} \mathrm{d}\tilde{\alpha} \, e^{i(\alpha+\tilde{\alpha})x} e^{i(\alpha-\tilde{\alpha})y} \rho^{i\alpha} \dot{K} \rho^{i\tilde{\alpha}} \frac{\operatorname{sech}(y) - 1}{y}. \tag{5.53}$$

The x integral gives $2\pi\delta(\alpha + \tilde{\alpha})$, so we end up with

$$a = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}\alpha \, e^{2i\alpha y} \rho^{i\alpha} \dot{K} \rho^{-i\alpha} \frac{\operatorname{sech}(y) - 1}{y}. \tag{5.54}$$

Thus, we have written a in terms of the modular flow of $\dot{K}$.

5.2.4 Coherent state path integral for Uhlmann phase

Let $\mathcal{M}$ be a space equipped with a continuous map $x \to |x\rangle$ from points $x \in \mathcal{M}$ to normalised states $|x\rangle \in \mathcal{H}$, and a measure $\mathrm{d}x$ such that

$$I = \int \mathrm{d}x \, |x\rangle \langle x| \tag{5.55}$$

is a resolution of the identity acting on $\mathcal{H}$. Such a construction is possible for any Hilbert space. The states $\{|x\rangle\}$ form an overcomplete basis for $\mathcal{H}$, and are sometimes referred to as coherent states. They are very useful for understanding classical limits, a topic we will have more to say about in Section 5.3.

By inserting the identity (5.55) twice in (5.32), one may write

$$e^{i\gamma} = \int \mathrm{d}x \int \mathrm{d}x' \, \langle x|\rho(0)|x'\rangle \, \langle x'|\mathrm{P}\exp\Big(\int_0^1 a(t)\,\mathrm{d}t\Big)|x\rangle \, . \tag{5.56}$$

Inserting the identity $n-1$ more times, the latter factor may be written

$$\langle x'|\mathrm{P}\exp\Big(\int_0^1 a(t)\,\mathrm{d}t\Big)|x\rangle = \lim_{n\to\infty} \langle x'|e^{\frac{1}{n}a(t_{n-1}) + \mathcal{O}\left(\frac{1}{n^2}\right)} \dots e^{\frac{1}{n}a(t_0) + \mathcal{O}\left(\frac{1}{n^2}\right)}|x\rangle \tag{5.57}$$

$$= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \mathrm{d}x_k \prod_{l=0}^{n-1} \langle x_{l+1}|e^{\frac{1}{n}a(t_l) + \mathcal{O}\left(\frac{1}{n^2}\right)}|x_l\rangle \, , \tag{5.58}$$

where $x = x_0$ and $x' = x_n$. As $n \to \infty$, in the usual way this expression becomes a path integral over continuous paths of coherent states $|x(t)\rangle$ obeying $|x_l\rangle = |x(t_l)\rangle$, which start at $|x\rangle = |x(0)\rangle$ and end at $|x'\rangle = |x(1)\rangle$. For such paths we have

$$\langle x_{l+1}|e^{\frac{1}{n}a(t_l)+\mathcal{O}\left(\frac{1}{n^2}\right)}|x_l\rangle = 1 + \frac{1}{n}\langle \dot{x}(t_l)|x(t_l)\rangle + \frac{1}{n}\langle x(t_l)|a(t_l)|x(t_l)\rangle + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{5.59}$$

$$= \exp\left(\frac{1}{n}\langle \dot{x}(t_l)|x(t_l)\rangle + \frac{1}{n}\langle x(t_l)|a(t_l)|x(t_l)\rangle + \mathcal{O}\left(\frac{1}{n^2}\right)\right). \tag{5.60}$$

Note that we are assuming that a/n is small enough that the above series expansions are valid – this can always be made true by taking n to be sufficiently large. (Just like in a usual path integral, the derivative of $|x(t)\rangle$ which appears here should be treated in a formal manner, because the paths which contribute to the path integral need not be differentiable.) Using this, we have

$$\langle x'|\mathrm{P}\exp\left(\int_0^1 a(t)\,\mathrm{d}t\right)|x\rangle$$
$$= \lim_{n\to\infty}\int\prod_{k=1}^{n-1}\mathrm{d}x_k \exp\left(\sum_{l=0}^{n-1}\left(\frac{1}{n}\langle \dot{x}(t_l)|x(t_l)\rangle + \frac{1}{n}\langle x(t_l)|a(t_l)|x(t_l)\rangle + \mathcal{O}\left(\frac{1}{n^2}\right)\right)\right) \tag{5.61}$$

After taking the limit, the sum converts to an integral, and we end up wih

$$\langle x'|\mathrm{P}\exp\left(\int_0^1 a(t)\,\mathrm{d}t\right)|x\rangle = \int \mathrm{D}x\,\exp(i\mathbf{S}[x]/\hbar). \tag{5.62}$$

where the action is

$$\mathbf{S}[x] = -i\hbar\int_0^1 \mathrm{d}t\,\Big(\langle\dot{x}|x\rangle + \langle x|a|x\rangle\Big). \tag{5.63}$$

The Uhlmann phase itself may then be written

$$e^{i\gamma} = \int \mathrm{D}x\ \langle x(0)|\rho(0)|x(1)\rangle\exp(i\mathbf{S}[x]/\hbar), \tag{5.64}$$

where now the path integral is done over all paths $x(t)$ (i.e. the endpoints $x(0)$ and $x(1)$ are integrated over also).

We should note for clarity at this point that the closed curve of density matrices $\rho(t)$ is independent of the path of coherent states $|x(t)\rangle$. The former is a fixed input in the computation, while the latter is a set of variables which are integrated over. In the path integral (5.64), we may view $\rho(t)$ as a fixed, time-dependent background field, while $|x(t)\rangle$ is a separate dynamical field.

5.2.5 Substituting in the Uhlmann connection

We now wish to substitute in our expression (5.54) for a. It is useful first to define some new states $|x(t,\alpha)\rangle$ by acting on the states $|x(t)\rangle$ with modular flow.

$$|x(t,\alpha)\rangle = \rho(t)^{-i\alpha}\,|x(t)\rangle\,. \tag{5.65}$$

We have written this in a deliberately suggestive way, putting t and α on the same footing in the left-hand side. This is because eventually these two parameters will both play the role of coordinates in the emergent bulk holographic spacetime. These states obey the 'modular Schrödinger equation'

$$\frac{\partial}{\partial\alpha}\left|x(t,\alpha)\right\rangle = iK(t)\left|x(t,\alpha)\right\rangle, \tag{5.66}$$

which implies that

$$\begin{aligned}\langle x(t,\alpha)|\dot{K}(t)|x(t,\alpha)\rangle &= \langle x(t,\alpha)|\left(\frac{\partial}{\partial t}(K(t)\left|x(t,\alpha)\right\rangle) - K(t)\frac{\partial}{\partial t}\left|x(t,\alpha)\right\rangle\right) && (5.67)\\ &= -i\left\langle x(t,\alpha)\right|\frac{\partial}{\partial t}\frac{\partial}{\partial\alpha}\left|x(t,\alpha)\right\rangle - i\left(\frac{\partial}{\partial\alpha}\left\langle x(t,\alpha)\right|\right)\frac{\partial}{\partial t}\left|x(t,\alpha)\right\rangle && (5.68)\\ &= -i\frac{\partial}{\partial\alpha}\left(\left\langle x(t,\alpha)\right|\frac{\partial}{\partial t}\left|x(t,\alpha)\right\rangle\right). && (5.69)\end{aligned}$$

Using this, we may write

$$\begin{aligned}\langle x(t)|a(t)|x(t)\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}\alpha\, e^{2i\alpha y}\,\langle x(t,\alpha)|\dot{K}(t)|x(t,\alpha)\rangle\,\frac{\mathrm{sech}(y)-1}{y} && (5.70)\\ &= -\frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}\alpha\, e^{2i\alpha y}\frac{\partial}{\partial\alpha}\left(\left\langle x(t,\alpha)\right|\frac{\partial}{\partial t}\left|x(t,\alpha)\right\rangle\right)\frac{\mathrm{sech}(y)-1}{y} && (5.71)\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}\alpha\, e^{2i\alpha y}\left\langle x(t,\alpha)\right|\frac{\partial}{\partial t}\left|x(t,\alpha)\right\rangle(1-\mathrm{sech}(y)) && (5.72)\\ &= \langle x(t)|\dot{x}(t)\rangle - \int_{-\infty}^{\infty}\mathrm{d}\alpha\,\mathrm{sech}(\pi\alpha)\left\langle x(t,\alpha)\right|\frac{\partial}{\partial t}\left|x(t,\alpha)\right\rangle. && (5.73)\end{aligned}$$

To get the first line, we just substituted in (5.54) and used the definition (5.65). To get the third line we integrated by parts with respect to α, and to get the fourth line we used the well-known Fourier transforms

$$\int_{-\infty}^{\infty} e^{2i\alpha y}\,\mathrm{d}y = \pi\delta(\alpha) \qquad \text{and} \qquad \int_{-\infty}^{\infty} e^{2i\alpha y}\,\mathrm{sech}(y) = \pi\,\mathrm{sech}(\pi\alpha). \tag{5.74}$$

Substituting this into (5.63), and noting that

$$\langle x(t)|\dot{x}(t)\rangle + \langle\dot{x}(t)|x(t)\rangle = \frac{\partial}{\partial t}\langle x(t)|x(t)\rangle = \frac{\partial}{\partial t}1 = 0, \tag{5.75}$$

one finds that the action for the Uhlmann phase may be written

$$\mathbf{S} = i\hbar\int_0^1\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\mathrm{sech}(\pi\alpha)\left\langle x(t,\alpha)\right|\frac{\partial}{\partial t}\left|x(t,\alpha)\right\rangle. \tag{5.76}$$

5.3 Classical limit

Let us now assume that the quantum system we are discussing has a classical limit in terms of coherent states. We should first explain what exactly this means. For more detailed information, see for example [159, 160].

Up to now, we have been acting as if there is a single fixed Hilbert space $\mathcal{H}$. However, there can actually be a different Hilbert space $\mathcal{H} = \mathcal{H}_\hbar$ for each allowed value of $\hbar$. Correspondingly, there is a map $x \mapsto |x\rangle$ for each value of $\hbar$ from $\mathcal{M}$ to $\mathcal{H}_\hbar$.

The space $\mathcal{M}$ should be thought of as containing all possible classical configurations, and the state $|x\rangle$ should be thought of as the quantum state which most closely approximates the classical state x, at a given value of $\hbar$.

By inserting the identity (5.55) many times, we can write the transition amplitude after a time T between coherent states $|x\rangle$ and $|x'\rangle$ as a path integral in the usual way, obtaining

$$\langle x'|e^{-iHT/\hbar}|x\rangle = \int \mathrm{D}x \exp(iS[x]/\hbar), \tag{5.77}$$

where the integral is done over paths $x(t)$ which begin at x and end at x', and the action is

$$S[x] = \int_0^T \Big(i\hbar \langle x|\dot{x}\rangle - \langle x|H|x\rangle\Big)\, \mathrm{d}t\,. \tag{5.78}$$

If we want there to be a good classical limit, then we need this action to be $\mathcal{O}(1)$ as $\hbar \to 0$, so that we can apply the usual methods of stationary phase.

The first term in $S[x]$ is proportional to the component of the Berry connection

$$i\,\langle x|\mathrm{d}|x\rangle \tag{5.79}$$

along the curve $|x(t)\rangle$. The Berry connection is a real 1-form on $\mathcal{M}$. For the action to be $\mathcal{O}(1)$, the Berry connection needs to be $\mathcal{O}(1/\hbar)$ as $\hbar \to 0$.

Operators can depend on $\hbar$, so when we talk about an 'operator', what we really mean is a family of operators, one acting on each Hilbert space $\mathcal{H}_\hbar$. The coherent states allow us to discuss the asymptotics of these operators as $\hbar \to 0$. In particular, when we write

$$O = \mathcal{O}(f(\hbar)) \tag{5.80}$$

for some function $f(\hbar)$, what we mean is that the coherent state correlators of O obey

$$\frac{\langle x_1|O|x_2\rangle}{\langle x_1|x_2\rangle} = \mathcal{O}(f(\hbar)) \quad \text{for all } x_1, x_2 \in \mathcal{M}. \tag{5.81}$$

A special case of this is $O = \mathcal{O}(1)$; an operator with this property is called a 'classical' operator. A requirement for the existence of a classical limit is that the Hamiltonian H is a classical operator. In [159], it is shown that $[O_1, O_2] = \mathcal{O}(\hbar)$ for any two classical operators O_1, O_2, and a corollary of this is that

$$e^{iO_1/\hbar} O_2 e^{-iO_1/\hbar} = \mathcal{O}(1). \tag{5.82}$$

So the automorphism generated by $iO_1/\hbar$ preserves the asymptotics of any operator.

By the requirements on the Berry connection and Hamiltonian, we have $S[x] = \mathcal{O}(1)$, so it can be treated as a classical action, and we have a good classical limit.

The functional (5.78) can be recognised as a classical Hamiltonian action if we identify

$$\Omega = \lim_{\hbar \to 0} i\hbar \, \mathrm{d} \langle x| \wedge \mathrm{d} |x\rangle \tag{5.83}$$

as the symplectic form, and

$$H(x) = \lim_{\hbar \to 0} \langle x|H|x\rangle \tag{5.84}$$

as the Hamiltonian function. By choosing canonical coordinates p_i, q_i on the space $\mathcal{M}$, we can write this (at least locally) as

$$\Omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i \qquad \text{and} \qquad H = H(p_i, q_i). \tag{5.85}$$

These canonical coordinates p_i, q_i represent the classical degrees of freedom.

Note that (5.78) naively appears to be a one-dimensional action. However, this formalism also describes the classical limit of higher-dimensional quantum field theories. In that case, the index i in p_i, q_i should be understood as containing continuous coordinates along space-like directions, and the $\sum_i$ includes an integration over those directions.

At this point, it may be tempting to naively repeat what we have just done for the Uhlmann phase action (5.63). Indeed, that action appears to be the same as (5.78), but with a Hamiltonian proportional to $\langle x|a|x\rangle$. So in a classical limit we might want to think of (5.63) as a classical Hamiltonian action with the usual symplectic form, but a different (and unusual) Hamiltonian function

$$\lim_{\hbar \to 0} i\hbar \langle x|a|x\rangle \,, \tag{5.86}$$

assuming the limit converges. The reason this is not correct is as follows. Within the Hamiltonian formalism, the Hamiltonian function *cannot* depend on the time derivatives of any classical variables, including background fields. Geometrically speaking, this is because it is just a function on phase space, and so it cannot know about derivatives along the particular trajectory on which we evaluate it. However, $\langle x|a|x\rangle$ depends on the time derivative $\dot{\rho}(t)$ of the background field $\rho(t)$. Hence, it would not be consistent to view (5.86) as a Hamiltonian function.

5.3.1 Highly entangled states

The conditions outlined above guarantee that the transition amplitude (5.77) has a good classical limit. However, they are not by themselves sufficient for the Uhlmann phase (5.64) to have a good classical limit. By a 'good classical limit', we mean that there is a single dominant contribution to the path integral, and that we can use a saddlepoint approximation.

Suppose that the modular Hamiltonian $K = -\log \rho$ obeys

$$K = \mathcal{O}\left(\frac{1}{\hbar}\right) \tag{5.87}$$

at each point $\rho = \rho(t)$ in the path along which we are computing the Uhlmann phase. This means that the states $|\Psi\rangle$ which purify ρ contain a large amount of entanglement in the classical limit.

Let us argue that (5.87) ensures that the Uhlmann phase has a good classical limit. First, consider what it implies about (5.54). By taking the time derivative of both sides of (5.87), we get $\dot{K} = \mathcal{O}(1/\hbar)$. Then, using (5.82) gives $\rho^{i\alpha}\dot{K}\rho^{-i\alpha} = \mathcal{O}(1/\hbar)$. Integrating over α does not change this scaling, so $a = \mathcal{O}(1/\hbar)$. Using this in (5.63), as well as the assumption that the Berry connection is $\mathcal{O}(1/\hbar)$, we see that $\mathbf{S}[x] = \mathcal{O}(1)$. Thus, as before, we can apply stationary phase methods to the path integral in the $\hbar \to 0$ limit.

Another perspective on this comes from the fact that (5.87) implies that modular flow reduces to a kind of classical evolution in the classical limit. Thus, the quantum states $|x(t,\alpha)\rangle$ correspond to classical states $x(t,\alpha) \in \mathcal{M}$. This means that the Berry connection along such states obeys

$$i \langle x(t,\alpha)| \frac{\partial}{\partial t} |x(t,\alpha)\rangle = \mathcal{O}(1/\hbar). \tag{5.88}$$

Thus, using (5.76), we see again that $\mathbf{S}[x] = \mathcal{O}(1)$.

5.3.2 Not very highly entangled states

Actually, there is a slightly more restricted regime we could consider, which is $K = o(1/\hbar)$. Then we would have $a = o(1/\hbar)$, and in this case the $\langle x|a|x\rangle$ term in the action (5.63) would go to zero in the classical limit. Thus, for the classical action we would only be left with

$$\mathbf{S}[x] = -i\hbar \int_0^1 \mathrm{d}t \, \langle \dot{x}|x\rangle \,. \tag{5.89}$$

Note that this is just the Berry phase of the path $|x(t)\rangle$.

However, the $\langle x|a|x\rangle$ term will be very important for the emergence of a holographic bulk. Thus, we will mainly consider the regime where $K = \mathcal{O}(1/\hbar)$.

5.3.3 The role of $\rho(0)$

The action $\mathbf{S}[x]$ is not the only contribution to the Uhlmann phase path integral (5.64); there is also the term

$$\langle x(0)|\rho(0)|x(1)\rangle \,. \tag{5.90}$$

We would like to understand what role this term plays when we take the classical limit.

The first thing we can say is that since this term only depends on the start $x(0)$ and end $x(1)$ of the curve $x(t)$, its effect can only be to set some initial and final boundary conditions on the classically dominant path – see for example the discussion in [116]. The evolution between $t = 0$ and $t = 1$ is thus still described only by the action $\mathbf{S}[x]$.

However, beyond this, the effect of this term will widely vary, depending on the exact nature of $\rho(0)$. One possibility is that (5.90) is sharply peaked around some specific classical state $\bar{x} = x(0) = x(1)$. Then in the classical limit this gives the boundary conditions. Alternatively, (5.90) may be peaked around a set of classical states $\{\bar{x}\}$. Then any classical limit would have to involve a sum over these boundary conditions.

Another interesting possibility is the case where $\rho(0)$ is a thermal density matrix. Then (5.90) can be computed using a Euclidean path integral. The total path integral (5.64) would then consist of two parts: the unitary Lorentzian evolution according to the action $\mathbf{S}$, and the thermal Euclidean evolution according to $\rho(0)$.

5.3.4 Symplectic form

At this point, we will make use of the rewriting of the Uhlmann phase action that we carried out above. In particular, we may recognize (5.76) as a classical Hamiltonian action. In this case, the Hamiltonian function vanishes, and the symplectic form is given by

$$\mathbf{\Omega} = \lim_{\hbar\to 0} i\hbar \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{sech}(\pi\alpha) \, \mathrm{d}\,\langle x(\alpha)| \wedge \mathrm{d}\,|x(\alpha)\rangle \,, \tag{5.91}$$

where $|x(\alpha)\rangle = \rho^{-i\alpha}\,|x\rangle$. Note that this symplectic form depends on the density matrix ρ.

It is useful to write this in terms of the canonical coordinates p_i, q_i on $\mathcal{M}$ that appeared in (5.85). First, let $p_{i,\alpha}, q_{i,\alpha}$ be the coordinates of the classical state $x(\alpha)$, so that the 1-parameter family of coordinates $p_{i,\alpha}, q_{i,\alpha}$ represents the classical evolution of p_i, q_i according to the classical limit of modular flow. Then we have

$$\mathbf{\Omega} = \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{sech}(\pi\alpha) \sum_i \mathrm{d}p_{i,\alpha} \wedge \mathrm{d}q_{i,\alpha} \,. \tag{5.92}$$

5.4 Holographic features

We can now justify the claim that the path integral for Uhlmann phase that we have derived is a holographic one. In order to be more precise, let us repeat here for convenience the action $S[x]$ for the transition amplitude, and the action $\mathbf{S}[x]$ for the Uhlmann phase:

$$S[x] = i\hbar \int_0^1 \mathrm{d}t \, \langle x| \frac{\partial}{\partial t} |x\rangle \,, \tag{5.93}$$

$$\mathbf{S}[x] = i\hbar \int_0^1 \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{sech}(\pi\alpha) \, \langle x(t,\alpha)| \frac{\partial}{\partial t} |x(t,\alpha)\rangle \,. \tag{5.94}$$

To avoid complicating things, we have turned off the Hamiltonian in $S[x]$.[2] Note that this means $S[x]$ is now just equal to the Berry phase along the path $x(t)$.

Our claim is that the action for the Uhlmann phase describes a theory living in one more dimension than the action for the transition amplitude. In an immediate sense this is clear from the forms of $S[x]$ and $\mathbf{S}[x]$: the latter involves an additional integration over an extra dimension, labelled by α. Because α parametrises modular flow, we can say that the additional dimension is generated by modular flow.

It is also worth comparing the symplectic forms for the two theories:

$$\Omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i \,, \tag{5.95}$$

$$\mathbf{\Omega} = \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{sech}(\pi\alpha) \sum_i \mathrm{d}p_{i,\alpha} \wedge \mathrm{d}q_{i,\alpha} \,. \tag{5.96}$$

We previously noted that $\sum_i$ can include an integral over spatial dimensions. The symplectic form for the Uhlmann phase similarly includes an integral over the extra dimension α. We can concisely write this if we define the combined index $I = (i, \alpha)$ and

$$\sum_I = \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{sech}(\pi\alpha) \sum_i \,. \tag{5.97}$$

Then we have

$$\mathbf{\Omega} = \sum_I \mathrm{d}p_I \wedge \mathrm{d}q_I \,. \tag{5.98}$$

The degrees of freedom $p_{i,\alpha}, q_{i,\alpha}$ at different values of α are not completely independent, but are related by modular flow. This is reminiscent of a gauge constraint, and supports the idea

[2] Alternatively, one could try to generalise the definition of Uhlmann phase so that a Hamiltonian plays a role in $\mathbf{S}[x]$. This is possible, and similar statements to those in this section can be made in that case. However, we will not make further reference to such a possibility in this thesis.

that the higher-dimensional bulk theory has gauge symmetry. To be more precise, consider the example of a Maxwell field A, for which the symplectic form is

$$\Omega_{\text{Maxwell}}(A, \delta_1 A, \delta_2 A) = \int_\Sigma \delta_1 A \wedge *\delta_2 F - \delta_2 A \wedge *\delta_1 F. \tag{5.99}$$

The components of $*F$ are not completely independent at different locations on the Cauchy surface, but are related by the Gauss constraint

$$\mathrm{d} * F\,\big|_\Sigma = 0. \tag{5.100}$$

In other theories with gauge symmetry, there are other similar constraints. This is completely analogous to what happens for the symplectic form $\boldsymbol{\Omega}$. In this case, the Gauss constraint is replaced by the modular Schrödinger equation (5.66).

We should note that if $K = o(1/\hbar)$, then (as described in Section 5.3.2) we actually have $S[x] = \mathbf{S}[x]$. So in that case, there is no emergent extra dimension. An extra dimension *only* emerges if $K = \mathcal{O}(1/\hbar)$ but $K \neq o(1/\hbar)$.

Beyond these quite general observations, there are a couple of concrete comparisons we can make with other known examples of holography, such as AdS/CFT. In that case, the modular Hamiltonian of a reduced typical state in a subregion is given at leading order by [98]

$$K = \frac{\hat{A}}{4G\hbar} + \dots. \tag{5.101}$$

Here $\hat{A}$ is an operator that measures the area of the HRT surface, and G is Newton's constant. This has the same scaling (5.87) that we have used in this chapter. Another feature of AdS/CFT is that one may use modular flow to reconstruct bulk operators [67]. This is reminiscent of, although perhaps not exactly the same as, the fact that modular flow generates the extra dimension parametrised by α.

One more point worth making might not provide any assistance with interpretation, but is simply interesting in its own right. In the case where $\rho(0)$ is a thermal state, the theory describing Uhlmann phase will be both d-dimensional *and* $(d+1)$-dimensional, where d is the dimension of the underlying theory! This is because the part of the action coming from the Lorentzian section of the evolution takes the same form as (5.94), whereas on the Euclidean section it takes the same form as (5.93) (with the Hamiltonian added back in, and continued to imaginary time).

Another way this change in dimension can happen is if along some sections of the curve $\rho(t)$, the modular Hamiltonian obeys $K(t) = o(1/\hbar)$, while on other sections it obeys $K(t) = \mathcal{O}(1/\hbar)$. When transitioning from the former sections to the latter, the extra dimension will 'condense' from the entanglement. When going from the latter sections to the former, the extra dimension 'dissolves' away again. It would be interesting to see how this relates to ideas in [1].

5.5 Discussion

We have demonstrated in this chapter that the Uhlmann phase of a generic system may be computed with a path integral. We have also shown that, if the state of the system is sufficiently highly entangled, a holographic higher-dimensional bulk appears to emerge in the classical limit of the path integral.

As we have tried to emphasise, the holographic interpretation we are espousing is an interesting possibility that we believe is worth exploring further, but it is at this point still fairly speculative. Let us now try to highlight some possible discomforts and open questions that will need to be addressed for the interpretation to become more concrete.

First, we have claimed that the classical limit of the action for the Uhlmann phase has an extra dimension, in the sense that it becomes a local action involving classical degrees of freedom labelled by $d+1$ coordinates, whereas the usual classical action involces classical degrees of freedom labelled by d coordinates. However, it may be possible that the appearance of the extra dimension is merely a fluke of notation. In particular, it may be possible that a clever rewriting of the action gets rid of the extra dimension. This rewriting would need to survive the classical limit (this is why we cannot just use (5.63) – there is no obvious way to take the classical limit of $\langle x|a|x\rangle$, without using the representation involving the α integral, and thus involving an extra dimension). We have not managed to find such a rewriting, and we strongly believe it is not generically possible – but we could of course be wrong. One way to show for certain that there is an extra dimension would be to exhibit degrees of freedom propagating in it. We have not attempted this.

A related complaint about our claim may be as follows: even if we can understand things in terms of an action with an extra dimension, the underlying theory doesn't have that extra dimension. So one might suggest that we are just expressing one path in terms of another – one path being the evolution of the extra-dimensional degrees of freedom, and the other being the evolution of the lower-dimensional degrees of freedom. Why not just use the lower-dimensional description of the underlying theory? To this, we would point out that the existence of dual descriptions is exactly the point of a holographic duality.

However, such a duality is only useful if it allows us to answer physical questions more easily.

In order for the duality we describe to do this, there needs to be some reason for the Uhlmann phase to be relevant to our questions. In particular, the evolution of the system we are interested in needs to obey the key conditions that define Uhlmann holonomy. It is not particularly clear that such a reason exists in general, as transition probabilities do not generally contribute to the evolution of the state. There are special situations where transition probabilities do contribute, for example if are classical observers involved, or if the system is undergoing decoherence. These would be worth exploring. Also, it may possible that transition probabilities play a role if the duality we are interested in involves an ensemble of theories. It has been suggested that some kinds of holographic duality have this property (e.g. [125]), so perhaps this is the best way to make progress.

An obvious question is how closely the holographic duality we have described is related to more widely-known examples of holography, such as AdS/CFT. Certainly the complementary nature of the results in this chapter and the preceding one are suggestive that there is such a relationship. However, the exact nature of this relationship remains mysterious.

Finally, there is a particular puzzle that is worth pointing out. Suppose we have a closed curve of states $|\psi(t)\rangle$ in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and let us assume that these states are highly entangled, so that $-\log \rho_A(t), -\log \rho_B(t) = \mathcal{O}(1/\hbar)$, where $\rho_A(t) = \mathrm{tr}_B \, |\psi(t)\rangle \langle\psi(t)|$ and $\rho_B(t) = \mathrm{tr}_B \, |\psi(t)\rangle \langle\psi(t)|$. According to the results of this chapter, the Uhlmann phases of the paths $\rho_A(t), \rho_B(t)$ may be computed in terms of a holographic bulk. According to the previous chapter, the same is true in AdS/CFT, i.e. when $\mathcal{H}_A \otimes \mathcal{H}_B$ is the Hilbert space of a holographic CFT, and A, B are boundary subregions. However, in AdS/CFT the holographic bulk also contributes to the Berry phase of the total pure state $|\psi(t)\rangle$, and it is not clear whether the same thing happens for the generic system we are considering in this chapter.

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